Riemann and Lebesgue Integration

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I. Introduction

As with other subject areas, mathematics contains a vast range of divisions, of which one is real Analysis. Analysis consists of the study of real and complex-valued functions. Within analysis lie the subfields: real analysis and complex analysis, which include the study of such topics as derivatives, integrals and series. For the following discussion on integration this paper is concerned only with the study of real-valued bounded functions, a subtopic of the real analysis field. Integration is by no means a new topic of discussion, but dates back to Archimedes in the 2nd century B.C. with his method of exhaustion. One of Archimedes' propositions provides an example of the method: "Similar polygons inscribed in circles are to one another as the squares on the diameters of the circles" (Kline 83). One would 'discover' the area of a circle by inscribing as many polygons, whose areas are known, as one can and proceed to sum the areas of those polygons. It was not until Newton and Leibniz's independent rigorous study of integration did the formal understanding and study of real analysis developed in the 17th century. It was their exhaustive efforts that eventually lead to Georg Friedrich Bernhard Riemann's and Henri Lebesgue's advancements in differential and integral calculus. By using the concept of measurable sets, the Lebesgue Integral provides a generalization of the Riemann Integral, applicable to a broadened range of functions than does the Riemann.

Bernhard Riemann (1826-1866), after deciding his true talent lay in mathematics and not theology, studied under German mathematicians Carl Friedrich Gauss and Wilhelm Weber at the University of Göttingen. Eventually Riemann, after Dirichlet's death, succeeded to a proper position as head professor of mathematics at Göttingen in 1859. Although this paper will only be discussing one of his contributions to real analysis, Riemann held a great passion for explaining the physical world. His key interests included "heat, light, the theory of gases, magnetism, fluid dynamics, and acoustics" (Kline 656). Most of his explanations and ideas came from his free use of "geometrical intuition and physical arguments," which is perhaps why he succeeded in his studies (Kline 656). Throughout his life Riemann's publications resulted in a combination of analysis with geometry, which provided a foundation for topology. Concerning integration, Riemann's Integral definition (by which he uses Riemann sums) impacted the rigorous study of real valued functions and is still used today.

Not long after Riemann's contributions, Henri Lebesgue (1875-1941) began his studies at the École Normale Supérieure, a new French school based on the teachings of the Enlightenment with a goal of providing France with "high-level professionals." Beginning with his paper concerning Weierstrass' approximation theorem, Lebesgue directly led himself to develop the foundations for measure theory, contained in his most famous work from 1902, "Intégrale, longueur, aire" ("Integral, Length, Area"). In the paper he outlines measure theory, defines integrals geometrically and analytically, and discusses Plateau's problem. Like Riemann, Lebesgue pursued multiple areas of research including integration, trigonometric functions and Fourier series, but his greatest contribution came from his theories on integration. Instead of limiting himself to using polygons, more specifically rectangles, for defining an integral, which focuses on the domain of a function, Lebesgue concerned himself with dividing a function into a set of measurable functions, later defined, that have to do with his unit of area, the measure. Out of this concern, Lebesgue developed the concept of Lebesgue measure, discussed later, which extended the realm of integrable functions beyond Riemann.

II. Necessary Framework

A. Set Theory

In order to begin studying integration one must first understand topics concerning set theory. As thought by Cantor, Frege and Russell, all of mathematics can be "based on set theory alone," thus why touching on some specific foundations is important (Royden 5). The most basic of foundations are sets; their properties determine the framework from which all functions, real numbers, etc. exist. The terms defined below will be used in the discussion of measure theory. Please refer to them as needed.

1. We will refer to elements of a <u>set</u> A as any object x belonging to A and write $x \in A$. Any set A is determined by its elements and we say $A = \{x : P(x)\}$, i.e. A is the set of all x from X such that x has the property P.
2. A collection of sets $\mathcal{C}$ is <u>countable</u> if there exists some injective mapping $q: \mathcal{C} \to \mathbf{N}$, where $\mathbf{N}$ is the natural numbers.
3. An <u>Algebra of Sets</u>: a collection $\mathcal{A}$ of subsets of $\mathbf{R}$ (the real numbers) that have the following properties,

   $(i)\ A \cup B \in \mathcal{A}$ *whenever* $A \in \mathcal{A}$ *and* $B \in \mathcal{A}$
   $(ii) \sim A \in \mathcal{A}$ *whenever* $A \in \mathcal{A}$
   $(iii)\ A \cap B \in \mathcal{A}$ *whenever* $A \in \mathcal{A}$ *and* $B \in \mathcal{A}$
4. An Algebra $\mathcal{A}$ of sets is called a <u>σ-algebra</u> if every union of a countable collection of sets in $\mathcal{A}$ is again in $\mathcal{A}$. That is, if $\{A_i\}$ is a sequence of sets, then $\bigcup_{i=1}^{\infty} A_i$ must again be in $\mathcal{A}$.

B. The Real Number System

Since we will be working with real valued functions, a review of terms is necessary for our discussions of the Riemann and Lebesgue integrals.

5. If S is a set of real numbers, we say b is an upper bound for S if for every $x \in S$, $x \leq b$. Furthermore, b is called the least upper bound (Sup) for S if it is the smallest upper bound for S. N.B. upper bounds need not belong to the set S and the least upper bound is unique.
6. Similarly, the lower bound a for a set S of real numbers is one in which for every $x \in S$, $a \leq x$. Also, a is called the greatest lower bound (Inf) for S if it is the largest lower bound for S.
7. Limit definition: We say $f(x) \to L$ as $x \to a$ if given any $\varepsilon > 0$, there exists a $\delta > 0$ such that $|f(x) - L| < \varepsilon$ whenever $0 < |x - a| < \delta$.
8. We say L is the limit of a sequence $\{a_n\}$ if for any $\varepsilon > 0$, there exists an N such that for every $n > N$, $|a_n - L| < \varepsilon$.
9. A sequence $\{a_n\}$ is said to converge if the limit exists and is said to diverge if the limit does not exist.
10. A sequence of functions $\{f_n\}$ is said to converge uniformly on an interval $[a, b]$ iff given $\varepsilon > 0$, there is an N such that for all $x \in [a, b]$ and all $n \geq N$ we have $|f_n(x) - f(x)| < \varepsilon$.

11. A function $f(x)$ is said to be continuous at a point a if and only if both $f(x)$ and $f(a)$ exist and $\lim_{x \to a} f(x) = f(a)$.
12. A function $f(x)$ is said to be continuous on a subset of $A \subset \mathbf{R}$ if it is continuous at each point of A.

13. A function $f(x)$ is said to be <u>bounded</u> if there exists an $M > 0$ such that $|f(x)| \leq M$ for every x in the domain.

14. <u>Completeness Axiom</u>: Every nonempty set S of real numbers which has an upper bound has a least upper bound.

III. Underlying Theory and Properties of Riemann and Lebesgue Integration

A. The Integral

What is an integral? In order for one to study its properties, applications and different forms, first one must define an integral. Webster defines integral as "made up, from, or formed of parts that constitute a unity," which is very close to the mathematical idea. An integral is the sum of partitions of a function. To obtain the general concept of an integral we may examine it through two different forms of presentation: constructive and descriptive. Once understanding an integral is achieved we may then proceed to its definition in the Riemann and Lebesgue senses.

There are three fundamental properties of the relationship between an antiderivative and its derivative:

a. *Lebesgue Normalizing Condition*

$$\int_a^b 1dx \ \ is \ \ (b-a).$$

b. If $F_1(x)$ and $F_2(x)$ are antiderivatives of $f_1(x)$ and $f_2(x)$ over the same interval, then $c_1F_1(x) + c_2F_2(x)$ is an antiderivative of $c_2f_1(x) + c_2f_2(x)$ over the same interval, for $c \in \mathbf{R}$.

c. If $F(x)$ is an antiderivative of $f(x)$ in $[a, b]$, and if $f(x)$ is non-negative in this interval, then $F(x) \geq F(a)$.

Now, we say that "the definite integral of $f(x)$ over an interval $[a, b]$, must be a real number that depends upon the values assumed by $f(x)$ in this interval" and the indefinite integral over $[a, b]$ is a real valued function satisfying the above properties a, b, c (Temple 22).

Secondly, we want to define the integral in descriptive language. As mentioned earlier, the constructive definition of area was arrived at from Archimedes and Eudoxus' method of exhaustion in which multiple polygons, whose areas are known, are inscribed into an unknown area in order to approximate the actual area. This is not dissimilar to Riemann's method of finding a definite integral over an interval I. Consider a bounded, positive function $f(x)$ (no need for $f(x)$ to be continuous) defined on the interval $[a, b]$ and the region R in the plane specified by

$$a \leq x \leq b \text{ and } 0 \leq y \leq f(x).$$

Divide the interval into a finite number of smaller intervals called <u>partitions</u> P, where

$P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ is a collection of $n + 1$ distinct points in $[a, b]$.

(There is no need for the points to be equally spaced).

Let

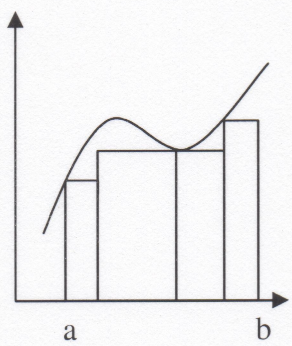
$$m_k = inf\{f(x) : x \in [x_{k-1}, x_k]\} \; and$$
$$M_k = \sup\{f(x) : x \in [x_{k-1}, x_k]\}.$$

The Lower sum of *f* corresponding to the partition is

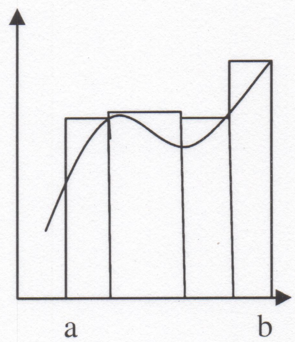
$$L = \sum_{k=1}^{n} m_k(x_{k-1} - x_k)$$

and the Upper sum of *f* corresponding to the partition is

$$U = \sum_{k=1}^{n} M_k (x_{k-1} - x_k)$$



Lower Sum



Upper Sum

Notice how the upper and lower sums are sums of areas of rectangles with base $[x_{k-1}, x_k]$ and height m_k and M_k, respectively. The following lemma is almost intuitive:

If $f: I \to \mathbb{R}$ is bounded, then for any partition on I, $L(f) \leq U(f)$.

The lower sums are always equal to or smaller than the upper sums; this holds true for every partition of $L(f)$ and $U(f)$.

Lastly, we are able to define the lower and upper integrals. The Lower integral of $f: I \to \mathbf{R}$ is

$$L(f) = sup\{L(P, f): P \in P(I)\},$$

and the Upper integral is

$$U(f) = inf\{U(P, f): P \in P(I)\}$$

where the sup/inf is taken over *all* the partitions of I. The upper and lower sums are able to be defined because of the completeness axiom (see page 4). Having defined an integral in descriptive and constructive terms we are ready to proceed onto Riemann's contribution to analysis.

B. The Riemann Integral

The Riemann integral is what one would encounter in any calculus based texts, such as those used by physicists or engineers. Riemann developed his definition for the integral as an attempt to define the term *area*. Although to the average person area seems an easy enough concept, to the mathematician *area* needs to be rigorously defined and have all its properties determined before he may consider using it continuously in proofs and problems. By dividing the area under the curve of a function into adjoining rectangular strips, as mentioned above, and summing the areas of each rectangle, one may find an estimate for the value of the definite integral over any defined interval. In order to find the exact area it is necessary to let the norm of P, defined as $\|P\| = max_{1\le i\le n}\Delta x_i$, tend toward 0:

$$\lim_{\|P\|\to 0}\sum_{k=1}^{n} m_k(x_{k-1} - x_k) .$$

As the width of the partitions approaches zero, the exact value for the integral emerges. Therefore, if the limits of the upper sums and the lowers sums converge to the same value L, then their common value is defined as the Riemann integral of f over the interval I. In other words:

If $L(f) = U(f)$, then we define the Riemann Integral to be this common value.

Let us denote the Riemann integral by:

$$R\int f(x)dx .$$

(Some of the many important properties of the Riemann integral will be discussed here and compared with those of the Lebesgue integral below in section IVA). Two questions arise when integrating a function: does the integral exist and if it does exist, how can it be evaluated? The answer to the first lies within logical reasoning; if $f: I \to \mathbf{R}$ is bounded on I, then f is integrable, in the Riemann sense, if and only if for any $\varepsilon > 0$, there exists some partition P of I such that the difference between the upper and lower sum is less than that ε. With notation:

$$\forall \varepsilon > 0, \exists\ P(I) \ni 0 < |U(f) - L(f)| < \varepsilon\ .$$

In other words, the error for the estimate of the area can be made arbitrarily small. The second questioned is answered in the above definition of the Riemann integral.

The following are some properties of the Riemann integral which will eventually be compared with Lebesgue's integral:

For integrable functions f and g:

15. Linearity

    If $k \in \mathbf{R}$, then the functions kf and $f + g$ are integrable on I and

    $$R\int kf = k\ R\int f$$

    $$R\int (f+g) = R\int f + R\int g\ .$$

16. If f and g are integrable on I and $f(x) \leq g(x)$ for all $x \in I$, then

    $$R\int f \leq R\int g$$

17. If $\{f_n\}$ is a sequence of functions that are integrable on I and the sequence converges uniformly on I to f, then

    $$\lim_{n\to\infty} R\int f_n(x)dx = R\int f(x)dx\ .$$

C. The Lebesgue Integral

Having already defined the integral in terms of Riemann sums, why is it necessary to define it in any different manner? In generalizing the integral one makes it applicable to an increased range of real valued bounded functions. Lebegue's method is based on the measure of sets. In addition, the extent of a function's discontinuities determines whether or not it is integrable, which raises the question: how can we measure the "length" of the set of discontinuities? Once knowing how to measure the length of discontinuities, how can we transform the Riemann integral into a summation of what we would later call *measurable* sets, instead of a summation of areas of rectangles? To answer both questions one needs to know something of Lebesgue's measure theory: what is a measurable set, a non-measurable set, and what is Lebesgue measure.

We will begin by defining the outer measure of a set. The first occurance of an outer measure came with Du Bois-Reymond's (1831-1889) *Die allgemeine Funktionentheorie*, some twenty years before Lebesgue generalized the measures of sets and applied measure theory to integration. Consider a set A covered with a countable collection of open intervals $\{I_i\}$; this then implies that $A \subset \cup I_i$.

1. The outer measure is the smallest sum of all the lengths of I_i's:

$$m^*A = \inf_{A \subset \cup I_n} \sum_{i=1}^{\infty} l(I_i).$$

As expected, the outer measure of any interval is its length. Also, if $\{A_i\}$ is a countable collection of sets of real numbers, then

$$m^* \cup A_n \leq \sum_{i=1}^{\infty} m^* A_i.$$

Following from the above proposition, if A is countable, then $m^*A = 0$.

1. A set E is said to be measurable if for any set A we have

   $m^*A = m^*A \cap E + m^*A \cap \sim E$

2. If a set is measurable, then the outer measure is the Lebesgue measure.

3. We shall say that m is a <u>countably additive measure</u> if it is a nonnegative extended real-valued function whose domain of definition is a σ -algebra $\mathfrak{M}$ of sets and we have

$$m(\bigcup E_n) = \sum mE_n$$ for each sequence $\{E_n\}$ of disjoint sets in $\mathfrak{M}$.

Outer measure is not countably additive, which is necessary for our goal to redefine the Riemann integral (a summation of areas). We have narrowed the family of sets for which outer measure is defined in definition 1 above; these are the types of sets we will deem measurable. (Royden 56). An important theorem follows: The collection $\mathcal{M}$ of measurable sets is a σ-algebra; i.e. the complement of a measurable set is measurable and the union (and intersection) of a countable collection of measurable sets is measurable. Moreover, every set with outer measure zero is measurable. (Royden 58).

Knowing measurable sets exist, does a non-measurable set exist? Yes. What makes Lebesgue's measure theory 'more complete' than those prior to his was (1) the addition of a set of measure zero, (2) the acknowledgment of the existence of non-measurable sets and 3) the notion of a measurable function. If you are curious, one such example of a non-measurable set is given on page 92 of Gelbaum and Olmstead's *Counterexamples in Analysis*. To integrate a function in the Lebesgue sense it must first be measurable (see definition of measurable function below) and secondly, have a measurable set over which to integrate. Therefore one need not worry about non-measurable sets because they will never be used for integrating real valued functions.

Having discussed measurable sets and defined Lebesgue measure, we are ready to begin generalizing the Riemann integral. First, it is necessary to determine what types of functions are deemed measurable.

Let E be a bounded measurable set on the x-axis. The function $f(x)$, defined for all points in E , is said to be <u>measurable on</u> E _if the set of points of E for which $f(x) > A$ is measurable for all values of A (Temple 104) .

It is easily shown that if $f(x)$ is an extended real-valued function whose domain is measurable, then the following are equivalent

1. For all $A \in \mathbb{R}, \{x : f(x) > A\}$ is measurable.
2. For all $A \in \mathbb{R}, \{x : f(x) \geq A\}$ is measurable.
3. For all $A \in \mathbb{R}, \{x : f(x) < A\}$ is measurable.
4. For all $A \in \mathbb{R}, \{x : f(x) \leq A\}$ is measurable.

The above four statements lead to the conclusion:

5. The set $\{x : f(x) = A, where\ A\ is\ any\ extened\ real\ number\}$ is measurable (Royden 65).

Therefore, by restricting our use to only measurable functions, the most important sets (above) connected with them are measurable (Royden 66). N.B. continuous functions are measurable and step functions are measurable. The following are desired theorems for measurable functions, (most of which seem obvious) since we want to work with these functions in the usual manner:

A. If f and g are each measurable in a set E , so also are the functions

$f + c, cf, f^2, f \pm g, fg, |f|$

where c is any real number.

B. If $\{f_n(x)\}$ is a sequence of functions measurable in E , then the functions

$\sup\{f_1, f_2, \ldots, f_n\}, \inf\{f_1, f_2, \ldots, f_n\}, \sup f_n, \inf f_n, \lim f_n(x), \lim f_n(x)$

are also measurable in E .

C. If f is a measurable function and $f = g$ almost everywhere, then g is measurable.

N.B. almost everywhere is defined as the set of points where $f \neq g$ has measure zero.

D. Let f be a measurable function defined on an interval I and assume that f takes the values $+\infty$ and $-\infty$ only on a set of measure zero. Then given $\varepsilon > 0$, we can find a step function g and a continuous function h such that

$|f - g| < \varepsilon \text{ and } |f - h| < \varepsilon$

except on a set of measure less than ε. If in addition f is bounded, then we may choose the functions g and h so that they are also bounded from below and above by the same values.

The sum, product, and difference of two simple functions are simple. (Royden 66-69)

In reading the theorems, one notices the similarity between measurable functions and the usual continuous real-valued functions and this is to be expected. Our goal is to generalize the Riemann integral, if the generalized functions possessed properties unlike Riemann integrable functions then the task at hand would be more difficult and would not be generalizing the Riemann integrable. The definitions of a characteristic and simple functions are needed in the definition of the Lebesgue integral. If A is any set, we define the characteristic function χ_A of the set A to be the function given by

$$\chi_A = \begin{cases} 1 \; if \; x \in A \\ 0 \; if \; x \notin A \end{cases}$$

A real valued function φ is called simple if it is measurable and assumes only a finite number of values. If φ is simple and has the values $a_1, \ldots, a_n$, then $\varphi = \sum_{i=1}^{n} a_i \chi_{A_i}$ where $A_i = \{x : \varphi(x) = a_i\}$.

The approach of Lebesgue integration is to begin by approximating $f(x)$ with a simple function. The integral of a function will be the sum of the measures of each measurable function, i.e. each step of the simple functions. A good starting point for creating an integrable function is the simple function $\varphi(x)$. If $\varphi(x)$ vanishes outside a set of finite measure then,

$$\int \varphi(x)dx = \sum_{i=1}^{n} a_i mA_i$$

where $\varphi = \sum_{i=1}^{n} a_i \chi_A$. In other words, the integral of φ is the sum of the a_i's multiplied by the measure of each A_i. Also, if E is any measurable set, we define

$$\int_E \varphi = \int \varphi \, \chi_E \, .$$

By multiplying φ by the characteristic function, we are transforming φ into an "on/off" function over the set E. The function is "on" whenever the characteristic function is 1, and the function is off whenever the characteristic function is 0. Now, if we let $\varphi(x) = \sum_{i=1}^{n} a_i \chi_{E_i}$ with each E_i disjoint from one another and each E_i being a measurable set of finite measure, then

$$\int_E \varphi = \sum_{i=1}^{n} a_i \, mE_i$$

One final definition is needed:

Let f be defined and bounded on a measurable set E with mE finite. It is necessary and sufficient that f be measurable, in order that

$$\inf \int_E \varphi = \sup \int_E \psi$$

Where the inf and sup over φ and ψ are simple functions in which $f \geq \phi$ and $f \leq \psi$.

Finally, if f is a bounded measurable function defined on a measurable set E with mE finite, we define the Lebesgue integral of f over E by

$$\int_E f(x) = inf \int_E \psi(x)$$

for all simple functions $\psi \geq f$. N.B. $\int_E f = \int_E f \chi_E$.

(Royden 75-77)

The Lebesgue integral definition generalizes the Riemann integral while still maintaining all aspects of the original integration. Proof of this fact is provided in the next section IV A. Notice this is a definition only for bounded measurable functions. The General Lebesgue Integral is defined as follows:

A measurable function f is said to be integrable over E if f^+ and f^- are both integrable over E. In this case we define

$$\int_E f = \int_E f^+ - \int_E f^-$$

where f^+ is defined as the positive part of the function f or $f^+(x) = \max\{f(x), 0\}$ and f^- is defined similarly as the negative part of the function f.

IV. Integrating Functions

A. Riemann versus Lebesgue

What makes a function integrable? When taking a calculus course students have the luxury of working with continuous functions that are always integrable, but of course there exist more than just continuous functions. A function must hold certain criteria to even consider its integrability over any interval.

The conditions for both Riemann and Lebesgue integrability are quite similar and in fact could be mistaken for being the same.

The function $f(x)$ is Riemann integrable iff the upper Riemann integral equals the lower Riemann integral:

$$R\int f(x)dx = \inf U = \sup L = R\int f(x)dx$$

Equivalently, if $f(x)$ is bounded and has set of discontinuities with measure 0, then $f(x)$ is Riemann integrable.

For the Lebesgue integral we have the following definitions, which are similar to Riemann.

1. Let f be defined and bounded on a measurable set E with mE finite. In order that

$$\inf_{f\leq\psi}\int_E \psi(x)dx = \sup_{f\geq\varphi}\int_E \varphi(x)dx$$

for all simple functions φ and ψ, it is necessary and sufficient that f be measurable. (Royden 77).

2. The definition of the Lebesgue Integral (see above).

The differences between the Riemann and Lebesgue construction lie within the summing over the intervals. Riemann sums over the domain of f, while Lebesgue sums over the range of f. Lebesgue has the added "requirements" that (1) the function be measurable and (2) the set over which it is integrated must also be measurable. Every bounded and finite function that is Riemann integrable is also measurable and, in this case, the Riemann integral is equal to the Lebesgue integral. The reason for why the two integrals are the same is that every step function is also a simple function and:

$$R\int_a^b f(x)dx \leq sup_{\varphi\leq f}\int_a^b \varphi(x)dx \leq inf_{\psi\geq f}\int_a^b \psi(x)dx \leq R\int_a^b f(x)dx$$

Since f is Riemann integrable, all the inequalities become equalities, and by (1) above f is measurable (Royden 79). We know f is Lebesgue measurable since $[a,b]$ is a measurable set of finite measure.

Any Riemann integrable function is Lebesgue integrable, but the converse statement does not necessarily hold true. One such example where this happens is the Dirichlet function:

$$D(x) = \begin{cases} 1 \; for \; x \in \mathbb{Q} \\ 0 \; for \; x \notin \mathbf{Q} \end{cases}$$

The Dirichlet function is discontinuous everywhere and therefore not Riemann integrable. For another way to show this without using the theorem consider the following:

Let $P = \{x_0, x_1, \ldots, x_n\}$ be an arbitrary partition of the interval $[0,1]$.

Between any two points x_i and x_{i+1} there is an irrational number.

$\Rightarrow inf_{[x_i, x_{i+1}]}$ must be 0.

$$\Rightarrow L(f,P) = 0$$

Similarly, between any two points x_i and x_{i+1} there is a rational number.

$\Rightarrow sup_{[x_i, x_{i+1}]}$ must be 1

$$\Rightarrow U(f,P) = (x_1 - x_0) + (x_2 - x_1) + \cdots + (x_n - x_{n-1}) = x_n - x_0 = 1 - 0 = 1$$

$\therefore L(f,P) \neq U(f,P)$ and the Dirichlet function is not Riemann integrable (Wachsmuth). ▪

On the other hand, the Dirichlet function is Lebesgue integrable as shown below.

Let $\mathbf{Q}$ be the set of all rational numbers, then the Dirichlet function restricted to the measurable set $[0,1]$ is the characteristic function of $A = \mathbb{Q} \cap [0,1]$. The set A is a subset of $\mathbf{Q}$, which means that A is measurable and $m(A) = 0$.

$$\Rightarrow \int_{[0,1]} D(x)\,dx = \int_{[0,1]} \chi_A(x)\,dx = m(A) = 0.$$

■

Unlike Riemann, a Lebesgue integrable function does not need to be continuous almost everywhere. Furthermore, Lebesgue showed that if the set of discontinuities of a function form a set of measure zero, then that function is Riemann integrable. I.e. if the set of discontinuities can be made so small as to not affect the integrability of a function, then it is indeed Riemann integrable. (Kline1046)

Next, we will compare the properties of the Riemann and Lebesgue integrals. Similarities occur with (1) multiplying by constant members of the real numbers, (2) adding and subtracting integrals of functions, (3) positivity of functions, (4) preservation of inequalities between functions, and (5) the monotone convergence theorem.

1. $\int kf = k\int f$
2. $\int f \pm g = \int f \pm \int g$
3. If $f \geq 0$, then $\int f \geq 0$
4. If $f \leq g$ for every x, then $\int f \leq \int g$ and $\left|\int f\right| \leq \int |f|$
5. Let $\{f_n\}$ be an increasing sequence of functions, and let $f = \lim_{n\to\infty} f_n$, then $\int f = \int \lim_{n\to\infty} f_n$.

All of the above hold true for all integrable functions f and g in the Riemann sense and all non-negative functions f and g over a measurable set E in the Lebesgue sense.

6. $R\int_a^b f = R\int_a^c f + R\int_c^b f$

Paralleling the above Riemann integral property:

7. Let f and g be integrable over E. If A and B are disjoint measurable sets contained in E, then $\int_{A\cup B} f = \int_A f + \int_B f$.

As one can see, Lebesgue generalizes the Riemann additive property because $[a, c]$ and $[c, b]$ are two measurable sets.

Differences occur in the slightest possible ways when generalizing the Lebesgue integral. The inequality property and the monotone convergence property slightly change to say: If f and g are nonnegative measurable functions, then as in (4) and (5) above

4. If $f \leq g$ almost everywhere, then $\int_E f \leq \int_E g$ and

5. Let g be integrable over E and let $\{f_n\}$ be an increasing sequence of functions such that $|f_n| \leq g$ on E and for almost all $x \in E$ we have $f = \lim_{n\to\infty} f_n$, then

$$\int_E f = \int_E \lim_{n\to\infty} f_n$$

Now we are able to say that the above five properties are true for all measurable functions.

B. Limitations of the Riemann Integral

As of yet it seems as though there is not much difference between Riemann and Lebesgue's integration techniques, except for the fact that Lebesgue uses measurable sets to integrate while Riemann uses rectangular partitions and the fact that Lebesgue integrates over a set A while Riemann integrates over an interval $[a, b]$. There are a number of differences that occur, mainly due to the fact that more functions are Lebesgue integrable than are Riemann integrable. For example, a function does not need to be continuous almost everywhere (as it does with Riemann) in order to be Lebesgue integrable. The following examples will illustrate the broadened range of functions which are Lebesgue integrable.

Example 1:

The Dirichlet function shown earlier in section IV A. ▪

Example 2:

Suppose $u_1(x), u_2(x), \dots$ are summable functions, meaning functions whose Lebesgue integral exists, on a measurable set E and the infinite sum $\sum_{n=1}^{\infty} u_n(x)$ converges to $f(x)$, i.e. we have a convergent series of summable functions. From this we see that $f(x)$ is measurable since $f(x)$ is the sum of measurable functions. If, in addition, the partial sum of the u_n's $= \sum_{n=1}^{n} u_n = S_n(x)$ is uniformly bounded, then it is a theorem that $f(x)$ is Lebesgue integrable on $[a, b]$ and

$$\int_a^b f(x)dx = \lim_{n\to\infty} S_n(x)dx$$

The difference lies here: if we wanted this to be true for Riemann integrals, then we would also need the fact that the sum of the series is integrable (Kline 1046). It is enough to know that each $u_n(x)$ is measurable, and therefore the partial sum results in a measurable function that is also Lebesgue integrable.

▪

Example 3:

One practical reason for increasing the amount of integrable functions is the ability to use Lebesgue's theories when dealing with Fourier Series. Riemann integration does not work kindly when taking limits of sequences of functions, which is what a Fourier Series is, and "Lebesgue integration is better able to describe how and when it is possible to take limits under the integral sign". The Fourier coefficients behave quite similarly to the Dirichlet function and by that fact alone one is able to see the advantages of using Lebesgue instead of Riemann.

Riemann stated that the Fourier coefficients a_n and b_n of a bounded and Riemann integrable function approach zero as n increases toward infinity. Lebesgue generalized:

$$\lim_{n\to\infty} \frac{1}{L}\int_{-L}^{L} f(x)\sin nx\,dx = 0$$

$$\lim_{n\to\infty} \frac{1}{L}\int_{-L}^{L} f(x)\cos nx\,dx = 0$$

for $n \in \mathbf{N}$.

Where $f(x)$ is any function, bounded or not, that is Lebesgue integrable. Of course, if $f(x)$ is integrable in the Riemann sense it is also integrable in the Lebesgue sense.

■

The Riemann integral allows for correspondence between the antiderivative and the derivative. When Riemann generalized his integral, a question of whether this fact held true for continuous functions in the more general case came to mind. One such example is the absolute value function $f(x) = |x|$. The function is indeed Riemann integrable, but its derivative does not exist at certain points, e.g. the point zero. Another example is any function that has finite removable or non-removable discontinuities at certain points; e.g. the greatest integer function

$(x) = [x]$; it is of course Riemann integrable, but it has no derivative or antiderivative at integer points.

From these few examples one sees how the Lebesgue integral is an improvement over the Riemann integral. There are many non-integrable functions in the Riemann sense in analysis; Lebesgue integration helps to solve some of their problems by presenting a different method in which these functions are integrable. Lebesgue preserves many of the aspects of the Riemann integral and, most importantly, agrees with the Riemann integral in all instances. The qualifications for the integrability of a function are similar in each sense, but different enough that Lebesgue's definition broadens the range of integrable functions. By integrating over measurable subsets of the range of a function, instead of over an interval in the domain, takes the focus away from the x-values and allows us instead to consider what is happening to the range of those x-values. Many of the conditions for theorems concerning Riemann integration are weakened when they are transformed into Lebesgue theorems. As shown by the few examples above, once certain conditions are weakened the set of functions which are integrable increases.

V. Conclusion

By using the concept of measurable sets, the Lebesgue Integral provides a generalization of the Riemann Integral, applicable to a broadened range of functions than does the Riemann integral. The background needed to understand the generalized integral is tedious but necessary. Before Lebesgue could generalize Riemann's integral he developed his set of measure theory, on which his integral is based, which includes the notion of a measurable set, a non-measurable set, a set with measure zero, outer and inner measure, Lebesgue measure, and, finally, measurable functions. By using measure theory to create newly generalized definitions and theorems, Lebesgue refined the Riemann integral and provided a platform for integration theory to continue.

Also necessary to understand before examining Lebesgue integration are set theory, the real numbers and the definition of the Riemann Integral. From this point, one uses the construction of the Riemann integral by finding upper and lower sums to find the value of an integral. (Equally, if one has enough knowledge and understanding of Lebesgue's measure theory, then instead he or she could have approached integration in the Lebesgue sense and not the Riemann sense, but this is not the case). Finally, one has the knowledge to construct the Lebesgue integral by dividing a function into a set of measurable functions and integrating over a measurable set.

Once the Lebesgue theory is developed one sees the advantages in using Lebesgue when integrating functions. The only drawback is that the functions must be bounded on a set of finite measure; there is no room for infinite integration. But even with this restriction the Lebesgue integral shows how the Riemann is restricted; when integrating sequences in the Lebesgue sense, "the generality and simplicity of the theorems of bounded and dominated convergence greatly facilitate the analysis" (Temple 174). For Riemann the sum of a series $\{f_n\}$ of functions must be integrable and the series must uniformly converge to f while Lebesgue does not need the restriction that the sum of the series be integrable.

The set of functions used in elementary calculus "is not 'closed' and most limiting processes take us out of these comfortable surrounding into a strange world of 'wild' functions where the elementary concepts of integration are no longer valid" (Temple 13-14). The specifics of functions, such as is the function monotone or continuous, are no longer of concern when using Lebesgue's theories; we may focus on the essential features of the problem. To review, Lebesgue proved that if $f(x)$ is bounded and measurable, then the indefinite Lebesgue integral

$$\Phi(x) = \int_a^x f(t)dt$$

possesses 'almost everywhere' a derivative $\Phi'(x)$ equal to $f(x)$. This allows $\Phi(x)$ to have no derivative at a set of points which has measure zero, the most significant improvement on Riemann's integral.

Why is Lebesgue so important to real analysis? His measure theory and integration techniques led to the method of monotone sequences invented by W.H. Young (1863-1942) expounded by L.C. Young (1905-2000), modification of the Darboux-Riemann method by Williamson (1897-1942), as well as the use of sequences of step functions, with some generalized type of convergence employed by Roger Ingleton (1920-2000). The Lebesgue and Riemann integrals are not the only types of integral existing; the list includes: Riemann-Stieltjies, Lebesgue-Stieltjies, Pettis, Bochner, Daniell, and Darboux. Each has its own unique differences and theorems but all are concerned with finding summations. By using Lebesgue's generalized definition of the integral, we can simplify integration theorems and types of integrable functions and make the entire concept of integration more applicable when conducting future research.

Works Cited

Allen, Don. "Riemann Integration." 8 March 2009 <http://www.math.tamu.edu/~dallen/history/riemann/riemann.html>.

Gelbaum, R. and Olmstead, John M.H. *Counterexamples in Analysis*. San Francisco: Holden Day, Inc. 1964

Kline, Morris. *Mathematical Thought from Ancient to Modern Times*. New York: Oxford University Press, 1972.

Larson, et.al. *Calculus with Analytic Geometry*. Boston: Houghton Mifflin Company, 2006.

McShane, Edward James. *Real Analysis*. New York: D. Van Nostrand Company, Inc., 1959.

Renze, John. "Analysis." < http://mathworld.wolfram.com/Analysis.html>.

Robert G., Bartle, et al. *Introduction to Real Analysis*. New York: John Wiley & Sons, 1982.

Royden, H.L. *Real Analysis*. New York: The Macmillan Company, 1968.

Temple, G. *The Structure of Lebesgue Integration Theory*. London: Oxford University Press, 1971.

Wikipedia. "Henri Lebesgue." 21 Feb. 2009 < http://en.wikipedia.org/wiki/Lebesgue>.

Wikipedia. "Bernhard Riemann." 21 Feb. 2009 < http://en.wikipedia.org/wiki/Riemann>.