Go Directly To Jail: The Mathematics Behind Family Game Night

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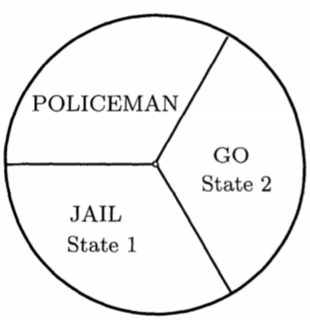
Senior Thesis

Family game night is fun, except when you are playing Monopoly with your family who happens to have a competitive streak. This pastime started with early board games, like Monopoly, and developed into the games we know and love today like Connect Four and The Game of Life. Board games are enjoyable to many people, but often players do not realize just how much mathematics is found in them. Overall, board game strategies are based in mathematics rather than just random chance with which they are usually associated. Thus, analyzing simple games, understanding sample mathematical scenarios, and further exploring Monopoly, will further the public's understanding of board games and their roots in mathematics. By looking at simple games, the processes and theorems are better understood, and by focusing on specific cases, like the Gambler's Ruin or Random Walk, will help to employ these concepts in a realistic scenario. Lastly, by focusing on Monopoly and comparing the expected value of specific properties and property combinations, we can discover the best choices in order to force your opponent into bankruptcy.

The mathematics behind games is something very intriguing. Some games clearly have basis in math, while others have it hidden. Games like the Tower of Hanoi have a mathematical basis but seem more like a strategy game, while others like Connect 4 have a quick way to win. Monopoly on the other hand has a multitude of variables that need to be considered to develop a winning strategy. There is diverse content in the large amount of articles that have been written exploring this popular game. I wanted to explore Monopoly to give myself a leg up on all my friends. Monopoly is a game of strategy that is based in many mathematical aspects. I have heard from many people that I have either played with or have watched play that the red properties are winning properties and Boardwalk is not good to have. I wanted to explore more in depth why this is so and the mathematical reasoning behind it.

A vital part of this research is understanding Markov chains and how they relate to board games. A Markov chain is a system with a finite number of states, where the chain moves from one state to another, and where the probability of moving from state *i* to state *j* is a number p_{ij} (Ross 185). In terms of a board game, state *i* is one space on the board, while state *j* is another and p_{ij} is the probability of moving from the first space, say "Go" in the case of Monopoly, and landing on the second space, like Community Chest. A vital part of my research involved transient and recurrent states of probability matrices. Transient implies that once in state *i,* a process enters state *i* a finite number of times. Recurrent implies that once in state *i,* the process will reenter this state infinitely, like the end of Chutes and Ladders. In other words, recurrent means that once in that state, a process will always return, while transient implies it might not return (Ross 195-196). Another important aspect of my research was the periods of states. Monopoly is ergodic, meaning it has positive recurrent, aperiodic states (Bilisoly). This also means that the time to get back to state *i,* or around the board, is finite and that there is no pattern of movement because of the random probabilities of the die (Ross 204).

To understand the process of deriving these probabilities, we want to use a simple example. This specific example can be found in *Take a Walk on the Boardwalk* by Stephen D. Abbott and Matt Richey. Let there exist a circle cut into three parts. Label these parts Jail, Go, and Policeman where Jail is state 1 and Go is state 2. In this example, the rules of the Policeman space are in effect and thus, the Policeman does not have a state because the player will never stay on this space. This is an easier method to calculate the probabilities as it takes out extra variables, but for later models, we will include it in our calculations. The player will



move clockwise by flipping a coin, obtaining either a 1 or a 2. Let the vector $x_i = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ be the probability that, after i rolls, the player is in state 1 or state 2. Thus, the column vector $x_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ represents that at the start of the game, a player will be located on Go. The next vector, $x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ represents that after one roll, the player will be in Jail with a probability of 1. After two rolls, the player has a ½ chance of ending up in Jail or on Go, thus $x_2 = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$. Thus, the probability matrix is $P = \begin{bmatrix} \frac{1}{2} & 1 \\ \frac{1}{2} & 0 \end{bmatrix}$.

The way to calculate the vector that explains the probability of landing in these states is best done by diagonalizing P. In this form, $P = NDN^{-1}$, where D is the diagonal matrix consisting of the eigenvalues of P and N is the matrix of the eigenvectors of P. Using Mathematica, we obtained $D = \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{2} \end{bmatrix}$ and $N = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}$. To calculate the vector after n rolls, the equation is $x_n = P.x_{n-1}$. In terms of calculating with x_0, the equation can be manipulated to create $x_n = P.x_{n-1} = P.(P.x_{n-2}) = \cdots = P^n x_0$ where $P = NDN^{-1}$. For example, to obtain x_{10}, the equation is,

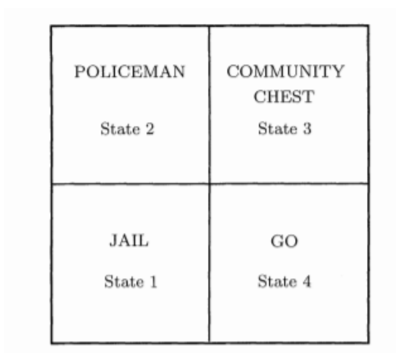
$$x_{10} = ND^{10}N^{-1}x_0 = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} . \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{2} \end{bmatrix}^{10} . \begin{bmatrix} \frac{1}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix} . \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{341}{512} \\ \frac{171}{512} \end{bmatrix}.$$

Thus, the probability of landing on Go after 10 rolls is $\frac{171}{512} \approx .334$ and on Jail is $\frac{341}{512} \approx .666$. This probability holds for any x_n where $n \geq 2$. There is a $\frac{1}{3}$ chance of being on Go and a $\frac{2}{3}$ chance of being in Jail. If we were to take the limit of x_n, we would obtain

$$\lim_{n\to\infty} P^n x = \lim_{n\to\infty} ND^n N^{-1} x = \begin{pmatrix} \frac{2}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{1}{3} \end{pmatrix} x = \begin{pmatrix} \frac{2}{3} \\ \frac{1}{3} \end{pmatrix}.$$

This concept is based on a theorem from Markov processes. To paraphrase this theorem, if P is an nxn transition matrix, there exists a vector v such that $Pv = v$. If x is a distribution vector, then $\lim_{n\to\infty} P^n x = v$ (Abbott 164). Therefore, $x_n = \begin{pmatrix} \frac{2}{3} \\ \frac{1}{3} \end{pmatrix}$ for all $n \geq 2$. This concept is vital in later methods to obtain the probabilities for Monopoly.

This process of diagonalization of the probability matrix works well only for small matrices, however, for a 4-space board, we run into some problems. The eigenvalues have an imaginary part, which makes it much more difficult to raise to a power and calculate x_n. In Mathematica, we can use numerical evaluation to the 3rd significant digit to work the complex part of these eigenvalues and eigenvectors into the value. Let us set up another example, specifically the 4-space board. Let there be a square split into four parts. There are 4 states: state 1 is Jail, state 2 is Policeman, state 3 is Community Chest, and state 4 is Go. Now, we will have three matrices that we want to multiply together. Let R be the transition matrix representing movement using a two-sided coin. Let J be the Jail transition matrix and let Cc be the Community Chest transition matrix. Thus,



$$R = \begin{bmatrix} 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}.$$

This means that there is a 50% chance of moving to Policeman or Community Chest when starting in Jail. This matrix forgoes the rule that Policeman sends you to Jail. This rule is represented in the Jail matrix, being

$$J = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The second column represents that when you land on Policeman, being state 2, you head straight to Jail, being state 1. The next matrix is the Community Chest matrix. Community Chest has 16 cards, two of which are included in this scenario. These cards are the one "Go to Jail" card and the one "Advance to Go" card. These are represented in this matrix,

$$Cc = \begin{bmatrix} 1 & 0 & \frac{1}{16} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{14}{16} & 0 \\ 0 & 0 & \frac{1}{16} & 1 \end{bmatrix}.$$

Now, we can use the same process above to solve for the probabilities in this scenario. We want to create one transition matrix though, so to do this, we will multiply the three above matrices together. Now, matrix multiplication is not commutative except when the matrices being multiplied are diagonalizable. This means that there exists a matrix Q such that for a matrix A, $Q^{-1}AQ$ is a diagonal matrix. The order of multiplication needs to represent what the player does. Since this example follows Abbott's 4-space example, the movement matrix is last because his matrices are transposes of mine. This means that while my state *i* is a row, his beginning state *i* is a column. This detail is a small difference between our methods, but both obtain accurate results. Normally, the player would first roll the dice to move, so the movement matrix would be first. Nevertheless, first we multiply the Community Chest and Jail Matrix. Now, note that $(Cc).(J) = (J).(Cc)$, so the order of Jail or Community Chest first does not matter. Then we multiply the movement matrix R last. Thus, the order of multiplication would be

$$(Cc).(J).(R) = \begin{bmatrix} \frac{17}{32} & \frac{1}{32} & \frac{1}{2} & 1 \\ 0 & 0 & 0 & 0 \\ \frac{7}{16} & \frac{7}{16} & 0 & 0 \\ \frac{1}{32} & \frac{17}{32} & \frac{1}{2} & 0 \end{bmatrix}.$$

This matrix can be used to solve for the probabilities, but it can be made simpler by removing the Policeman state as a player never spends a turn on it. Thus, we can remove the second row and column as it has no effect on the final probabilities. Our new matrix is

$$K = \begin{bmatrix} \frac{17}{32} & \frac{1}{2} & 1 \\ \frac{7}{16} & 0 & 0 \\ \frac{1}{32} & \frac{1}{2} & 0 \end{bmatrix}.$$

Now, by following the same process that we used for a 3-space scenario, we find that

$$x_n = \begin{bmatrix} \frac{16}{27} \\ \frac{7}{27} \\ \frac{4}{27} \end{bmatrix}.$$

Thus, the probability of landing in Jail is about 0.593, the probability of landing on Community Chest is 0.259, and lastly, the probability of landing on Go is 0.148 (Abbott).

Since we have worked through the process of calculating these probabilities in simpler scenarios, we can now work our way up to Monopoly and the many rules that we need to consider. The rules of Monopoly are important to understanding the models created in this paper. First, players start on Go. They roll two 6-sided dice in order to move. If a player lands on Chance, they pull a card from the Chance pile. Once a card is pulled, it is shuffled back into the pile. If a player lands on Community Chest, they pull a card from the Community Chest pile. This card also is shuffled back into the deck. If a player lands on Free Parking, they gain nothing. If a player lands on the Policeman, they go directly to jail. These rules will be used in all models, but rules regarding Jail will be added later. Since a player will be rolling two dice, the probabilities are as follows:

Value	**Probability**	**Value**	**Probability**
2	$\frac{1}{36}$	8	$\frac{5}{36}$
3	$\frac{2}{36}$	9	$\frac{4}{36}$
4	$\frac{3}{36}$	10	$\frac{3}{36}$
5	$\frac{4}{36}$	11	$\frac{2}{36}$
6	$\frac{5}{36}$	12	$\frac{1}{36}$
7	$\frac{6}{36}$		

First, we will start with the base of 39 states for a $40x40$ matrix board. This example is most similar to what Roger Bilisoly created in his paper "Using Board Games and Mathematica to Teach Fundamentals of Finite Stationary Markov Chains" as he had 39 states and did not include the rules regarding Jail. However, my model is slightly different than his as he used

Mathematica code to create his matrices and find the eigenvalues. In this examination of the game Monopoly, assume Go is state 0 and Boardwalk is state 39. First, we will focus on the movement matrix. Each row contains this pattern of probabilities. For example, at state 0, the row is:

$$\begin{pmatrix} 0 & 0 & \frac{1}{36} & \frac{2}{36} & \frac{3}{36} & \frac{4}{36} & \frac{5}{36} & \frac{6}{36} & \frac{5}{36} & \frac{4}{36} & \frac{3}{36} & \frac{2}{36} & \frac{1}{36} & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

This means that when a player starts on Go, they have zero chance to move back to go as they must move forward. They also have zero chance going one space forward as it is impossible to roll a 1 on two 6-sided dice. However, a player has a $\frac{1}{36}$ chance of moving two spaces forward, being snake eyes. Every row of the movement matrix has this pattern, just at a different location depending on the space. The pattern moves one space over each row and wraps back around to the beginning at the 28th state. Similarly to our analysis of the 4-space board, we assume that you can land and move from Policeman or state 30.

Now we will create the Jail matrix. For this matrix, the diagonal is filled with ones except for in the 30th state, where the one is placed in the 10th state (shown below):

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

This format means that if you are on Go, you will stay on Go because you cannot possibly go to Jail from that point. This matrix ultimately represents spaces on the board that send you to Jail just by landing on said space. Community Chest and Chance are not included in this because by landing on those spaces, you will not be sent to Jail immediately and with 100% certainty.

The next matrix would be the Community Chest matrix. Community Chest has 16 cards, two of which send you to either Go or to Jail. Thus, the matrix has ones in the diagonal, except in the 3rd, 18th, and 34th row. These rows have $\frac{14}{16}$ in the diagonal and $\frac{1}{16}$ in the 0 and 10th state. An example of this is shown below, pulled from the 2nd state of this matrix:

$$(\frac{1}{16} \quad 0 \quad \frac{14}{16} \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad \frac{1}{16} \quad 0 \quad 0 \quad 0 \quad 0 \quad \cdots \quad 0 \quad 0).$$

Lastly, we will create a $40x40$ Chance matrix. Chance has many more cards that will send the player to a different space. The probabilities are as follows:

Card	**Probability**
Advance to Go	1/16
Go back 3 Spaces	1/16
Take a Trip to Reading Railroad	1/16
Advance to the nearest Railroad	2/16
Go Directly to Jail	1/16
Advance to St. Charles	1/16
Advance to the Nearest Utility	1/16
Advance to Illinois Ave	1/16
Advance to Boardwalk	1/16

Thus, each Chance has a slightly different format based on location. For example, the first Chance located in the 7th state is shown below:

$$(\frac{1}{16} \quad 0 \quad 0 \quad 0 \quad \frac{1}{16} \quad \frac{3}{16} \quad 0 \quad \frac{6}{16} \quad 0 \quad 0 \quad \frac{1}{16} \quad \frac{1}{16} \quad \frac{1}{16} \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1/16 \quad 0 \quad 0 \quad \cdots \quad 0 \quad \frac{1}{16})$$

This can be interpreted as there is a $\frac{1}{16}$ chance of going back to Go, $\frac{1}{16}$ chance of moving back three spaces, $\frac{3}{16}$ chance of moving to Reading Railroad, and so on following the chart of Chance probabilities. Each row is slightly different based on location as some cards say to go to the nearest railroad or utility, meaning that the location of Chance will change which space has a probability of being landed on. For example, the first chance at the 7th space has a $\frac{3}{16}$ probability of moving to Reading Railroad because there are two cards that say move to the nearest railroad and one that says to move to Reading Railroad.

Now that we have created these matrices, it is time to consider how they are being multiplied and the results obtained. Since matrix multiplication is not commutative, then we first need to consider which matrix is acted on first. The player would first roll their dice and thus, the movement matrix must be multiplied first. The next matrix being multiplied depends on where a player lands. As such, the order of the Jail, Community Chest, and Chance does not matter. This fact holds for all models I created. While our process outlined above works for small matrices, I discovered that this process failed for larger ones. My computer could produce the $40x40$ matrix's eigenvalues, but it was very difficult to generate the eigenvectors. Because the eigenvalues were larger than the smaller models, calculating the eigenvectors through Mathematica was much more time consuming. My computer could not complete this process, so I worked to find a new process to approximate these probabilities.

By raising a matrix to a large power, we approximate the long-term probabilities instead of the exact values through diagonalization. Thus, we can raise a matrix to a larger power, such as 100, and multiply it by a state vector with the same number of columns to obtain these probabilities. For this $40x40$ matrix, we multiply $(T).(J).(Ch).(Cc)$ and raise it to the 100th power. Then we multiply it by the state vector, a 40 column and 1 row vector with 1 in the first column. In order to get readable output from Mathematica, we round the probabilities we obtain to the three significant digits, which made the calculation process and analysis easier. Therefore, we can evaluate numerically in Mathematica to obtain these probabilities:

{{0.0314, 0.0218, 0.0194, 0.0222, 0.0239, 0.0303, 0.0232, 0.00887, 0.0237, 0.0236, 0.0590, 0.0277, 0.0252, 0.0241, 0.0249, 0.0264, 0.0279, 0.0257, 0.0290, 0.0304, 0.0284, 0.0279, 0.0103, 0.0270, 0.0315, 0.0302, 0.0267, 0.0265, 0.0292, 0.0256, 0, 0.0266, 0.0261, 0.0236, 0.0250, 0.0273, 0.00869, 0.0220, 0.0220, 0.0266}}

The first value of 0.0314 is the proportion of time spent on Go, the next is the probability of landing on Mediterranean Avenue. The value I want to bring your attention to, however, is that of the Jail probability, being 0.0590. This means in this model of Monopoly, the player has a 5.9% chance of going to Jail. This makes sense as there are many different ways to get to Jail: by landing on it, by pulling a card from Chance or Community Chest, or landing on the Policeman. These probabilities are similar to those that Roger Bilisoly obtained. I believe the difference between them is simply different rounding between these models. This is the beginning of modeling the game Monopoly in order to discover which spots are landed on the most.

In the $40x40$ matrix model, we did not consider some rules of the game that are normally included. For sake of simplicity, some papers, like Roger Bilisoly's, did not include the rules regarding Jail, being that by rolling three doubles in a row the player goes to Jail and that if the player rolls a double while in Jail, they are let out. However, these rules affect the probabilities of spaces being landed on. Thus, the next model I created was a $41x41$ matrix, where state 10 was visiting Jail and state 41 was Incarcerated in Jail. This model is similar to the previous one, each matrix is the same except they have another row and column. For the movement matrix, the 41st column is filled with zeros as it is impossible to move from a space on the board to Incarcerated in Jail with the rules we have included in this model. We exclude the rule regarding going to Jail after rolling three doubles but will consider it in a later model. However, we assume that a player must leave Jail after the first turn in Jail. Thus, a player must leave Jail the following turn after they are incarcerated. The below figure represents our new movement matrix including this assumption about leaving Jail:

$$\begin{pmatrix} 0 & 0 & \frac{1}{36} & \frac{2}{36} & \frac{3}{36} & \frac{4}{36} & \frac{5}{36} & \frac{6}{36} & \frac{5}{36} & \frac{4}{36} & \frac{3}{36} & \frac{2}{36} & \frac{1}{36} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \frac{1}{36} & \frac{2}{36} & \frac{3}{36} & \frac{4}{36} & \frac{5}{36} & \frac{6}{36} & \frac{5}{36} & \frac{4}{36} & \frac{3}{36} & \frac{2}{36} & \frac{1}{36} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{36} & \frac{2}{36} & \frac{3}{36} & \frac{4}{36} & \frac{5}{36} & \frac{6}{36} & \frac{5}{36} & \frac{4}{36} & \frac{3}{36} & \frac{2}{36} & \frac{1}{36} & \cdots & 0 \end{pmatrix}.$$

The Jail, Community Chest, and Chance matrix in this model are similar to that of the $40x40$ model. For the Jail Matrix, the row representing the policeman has a one in the last column, representing Incarcerated in Jail. Likewise, the Chance and Community Chest matrices are the same except the $\frac{1}{16}$ chance to go to Jail in now in the last column of the matrices in their respective rows. Now, we will multiply these together, raise it to a power, and then multiply it by the state vector. Thus, our equation is $P = State.(T.J.Ch.Cc)^{100}$. The resulting probabilities as calculated in Mathematica are displayed below:

{{0.0314, 0.0218, 0.0194, 0.0222, 0.0239, 0.0303, 0.0232, 0.00887, 0.0237, 0.0236, **0.0233**, 0.0277, 0.0252, 0.0241, 0.0249, 0.0264, 0.0279, 0.0257, 0.0290, 0.0304, 0.0284, 0.0279, 0.0103, 0.0270, 0.0315, 0.0302, 0.0267, 0.0265, 0.0292, 0.0256, 0, 0.0266, 0.0261, 0.0236, 0.0250, 0.0273, 0.00869, 0.0220, 0.0220, 0.0266, **0.0357**}}

The first value is the proportion of time spent on Go, the next is the probability of landing on Mediterranean Avenue, and so on. These probabilities are the same as those created by the $40x40$ model. The only difference between those two is that the Jail probability we obtained from the $40x40$ model is now split in this model. By adding the bolded values, we obtain the frequency of landing on Jail in Monopoly, being $0.0233 + 0.0357 = 0.0590$. This value matches with the probability from the $40x40$ model. This makes sense as we did not add any rules in the $41x41$ model that would affect how often any one space is landed on compared to the $40x40$ model. Because of this result, we now want to examine Monopoly with more rules and more states.

Now with both models completed, I decided to examine a $43x43$ model with more rules included. Recall that the states in this model are state 0 is Go, state 1 is Mediterranean Ave, ..., state 10 is Visiting Jail, ..., state 39 is Boardwalk, state 40 is Behind Bars, state 41 is First Turn in Jail (No doubles rolled), and state 42 is Second Turn in Jail (No doubles rolled). Using this model, we include rules that the paper "Using Board Games and Mathematica to Teach the Fundamentals of Finite Stationary Markov Chains" by Roger Bilisoly did not. First, we will assume that after rolling three doubles in a row, a player will be behind bars. Secondly, if a player rolls doubles, they must leave Jail and travel the amount of spaces which they rolled. Lastly, on a player's third turn, they will be forced to pay the fine of $50 and leave jail. In the following pages, I will outline how each matrix is set up. I will include a copy of each matrix in the Appendix.

First, we will examine the movement matrix. The first 40 rows by 40 columns are the same as the movement matrix used in the $40x40$ model. Recall the following probability table:

Value	**Probability**	**Value**	**Probability**
2	$\frac{1}{36}$	8	$\frac{5}{36}$
3	$\frac{2}{36}$	9	$\frac{4}{36}$
4	$\frac{3}{36}$	10	$\frac{3}{36}$

5	$\frac{4}{36}$	11	$\frac{2}{36}$
6	$\frac{5}{36}$	12	$\frac{1}{36}$
7	$\frac{6}{36}$		

Each row contains these probabilities and they are located two spaces from the diagonal in each row, meaning in state 10, the probability of rolling a two is located in the 13th column. Once the matrix reaches the lower part of the board, these values begin to wrap back around to the beginning of the row. Thus, in state 37, there is a $\frac{2}{36}$ chance of landing on Go from Park Place. The whole matrix is located in the Appendix.

The last three states model the rules of exiting Jail. In state 40 and 41, a player has a $\frac{1}{36}$ chance of rolling doubles of any one value. The rules of Monopoly state that if a player in Jail rolls a double, they must exit Jail and move forward the number of spaces that they rolled. Thus, if a player rolls doubles, they will move from state 10, being Visiting Jail. This is due to the fact that while this model has added states for behind bars, rolling doubles and getting out of jail would move the player forward from the space that Jail is located. Since Visiting Jail is state 10, then if a player rolls doubles, they move from that state, as shown below:

Spaces from Jail	0	1	2	3	4	5	6	7	8	9	10	11	12
Probability of Leaving after the First Turn in Jail	0	0	$\frac{1}{36}$	0	$\frac{1}{36}$	0	$\frac{1}{36}$	0	$\frac{1}{36}$	0	$\frac{1}{36}$	0	$\frac{1}{36}$
Probability of Leaving after the Second Turn in Jail	0	0	$\frac{1}{36}$	0	$\frac{1}{36}$	0	$\frac{1}{36}$	0	$\frac{1}{36}$	0	$\frac{1}{36}$	0	$\frac{1}{36}$

The player has a $\frac{30}{36}$ chance of staying behind bars, meaning that the probability of going from state 40 to state 41 is $\frac{30}{36}$. This is because state 40 is when the player is behind bars and if the player does not roll doubles, then they go into their first turn behind bars meaning they move into state 41. Similarly, the probability of going from state 41 to state 42 is also $\frac{30}{36}$ by the same reason. Now, on the players third turn in Jail, they must pay the fine and leave. As such, the players will move from state 10 with the probabilities from rolling two dice, similarly to the pattern from the rest of the matrix. Thus, the probabilities of moving from state 42 to the rest of the board is as follows:

$$(0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad \frac{1}{36} \quad \frac{2}{36} \quad \frac{3}{36} \quad \frac{4}{36} \quad \frac{5}{36} \quad \frac{6}{36} \quad \frac{5}{36} \quad \frac{4}{36} \quad \frac{3}{36} \quad \frac{2}{36} \quad \frac{1}{36} \quad 0 \quad \cdots \quad 0)$$

This describes the $43x43$ movement matrix, Figure 2 in the Appendix, modeling the rules of exiting Jail in Monopoly.

Next, we shall work through the Jail matrix. Due to the addition of two new states and the rule of rolling doubles, this matrix has become more complex. Since the probability of rolling a double is $\frac{6}{36}$, the probability of rolling three doubles in a row is $(\frac{6}{36})^3$. This is because the probability of rolling a double does not change if you already rolled one, meaning it is independent. Then, we obtain that the chance of rolling three doubles in a row is $\left(\frac{6}{36}\right)^3 = \frac{1}{6} * \frac{1}{6} * \frac{1}{6} = \frac{1}{216}$. Recall in the Jail matrix for the $40x40$ model, the diagonal was filled with ones except on state 30 which is the Policeman space. This $43x43$ matrix is similar, except that the probability of staying in each state is $\frac{215}{216}$ and the probability of going to state 40, or behind bars, is $\frac{1}{216}$. We assume that at any point on the board, a player can go to Jail due to the double rule. The exception to this format is state 30, where a player will go to state 40 with 100% certainty. Also, if a player is in state 40, 41, or 42, they will stay in that state. Thus, we have formatted a Jail matrix with the rule of rolling doubles three times. It should be noted that this method only addresses the rules of doubles in regards to the Jail space rather than all the spaces, but it does create accurate probabilites for our end result.

The next matrix we will model will be the Community Chest matrix. This matrix is by far the easiest out of the four. This matrix has ones down the diagonal except in states 2, 17, and 33 where the diagonal is $\frac{14}{16}$. This is because there are 14 cards in Community Chest that will not move the player, and there are two that do- one being "Advance to Go" and the other being "Go to Jail." These cards will send a player to state 0 and state 40 respectively. Therefore, we have created the Community Chest matrix as shown by Figure 4 in the Appendix.

The Chance matrix for this $43x43$ model, Figure 5 in the Appendix, is similar to that which we already worked through in the $40x40$ model. The only difference with this matrix is that there is a $\frac{1}{16}$ chance the player ends up behind bars, which is state 40, by the "Go to Jail" card. Even though state 41 and 42 are also technically Jail states, the player cannot enter these states through the Chance matrix as these states are after a player has spent a turn in Jail. I have included these matrices in the Appendix in order to clear up any confusion.

Finally, we can calculate the probabilities of landing on Monopoly spaces. With these matrices, we will first multiply them together. This is done the same way as the 40x40 model, where we will first multiply the movement matrix by the Jail matrix. Then we multiply by the Chance matrix and Community Chest matrix in that order. By following the new process to approximate diagonalization, we will raise this product to the power of 100 and then multiply it by the state vector as shown below:

$$(1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0)$$

.

To obtain these probabilities, we use the equation $P = x_0.(T.J.Ch.Cc)^{100}$ where x_0 is the State vector. Therefore, our probabilites are:

State	Space Name	Probability (%)
0	Go	2.93
1	Mediterranean Ave	2.03
2	Community Chest	1.80
3	Baltic Ave	2.07
4	Income Tax	2.22
5	Reading Railroad	2.83
6	Oriental Ave	2.15
7	Chance	.824
8	Vermont Ave	2.20
9	Connecticut Ave	2.19
10	Visiting Jail	2.16
11	St. Charles Place	2.58
12	Electric Company	2.50
13	States Ave	2.20
14	Virginia Ave	2.44
15	Pennsylvania Railroad	2.38
16	St. James Place	2.69
17	Community Chest	2.30
18	Tennessee Ave	2.81
19	New York Ave	2.78
20	Free Parking	2.80
21	Kentucky Ave	2.57

State	Space Name	Probability (%)
22	Chance	1.03
23	Indiana Ave	2.52
24	Illinois Ave	2.95
25	B. & O. Railroad	2.85
26	Atlantic Ave	2.50
27	Ventnor Ave	2.48
28	Water Works	2.76
29	Marvin Gardens	2.40
30	Policeman	0
31	Pacific Ave	2.49
32	North Carolina Ave	2.45
33	Community Chest	2.21
34	Pennsylvania Ave	2.34
35	Short Line	2.55
36	Chance	.813
37	Park Place	2.06
38	Luxury Tax	2.05
39	Boardwalk	2.49
40	Behind Bars	3.78
41	Behind Bars (First Turn)	3.15
42	Behind Bars (Second Turn)	2.63

This chart represents the probability a player will land on a specific property. In this model I have created, we have another Jail state compared to "Take a Walk on Boardwalk" by Stephen Abbott and Matt Richey. Our calculated data states that there is a 11.72% chance that the player will end up in Jail, whether that be visiting Jail or behind bars. This is the same probability as Abbott's article. Since I added the third Jail state, this probability should be larger. I believe the cause of this result is most likely due to the rounding. When we try more decimal places, we obtain the following probabilities:

{{0.029282, 0.020336, 0.018038, 0.020709, 0.022225, 0.028306, 0.021539, 0.0082409, 0.022035, 0.021903, 0.021608, 0.025784, 0.025033, 0.021976, 0.024440, 0.023781, 0.026915, 0.022952, 0.028095, 0.027826, 0.027956, 0.025745, 0.010275, 0.025222, 0.029522, 0.028507, 0.025037, 0.024850, 0.027551, 0.024040, 0, 0.024931, 0.024473, 0.022095, 0.023387, 0.025534, 0.0081264, 0.020590, 0.020550, 0.024929, 0.037842, 0.031535, 0.026279}}.

By rounding to the 5^{th} significant digit, the probability a player will land on the Jail space is 11.7559%. Thus, by adding another Jail state, our probability of landing on Jail has increased compared to the result obtained by Abbott.

In Abbott's article, which contained the data that I used as a comparison point to my results, the next most landed on property was Illinois Avenue, which matches with my findings. This relates back to the fact that many say that the red properties are good to own. However, Illinois is the only red with that high of a probability as Indiana and Kentucky are in the top 15, being number 14 and 12 respectively. Curiously, the orange properties, which are approximately 7 spaces away from Jail (an average value for 2 die), were landed on less than Illinois Avenue. However, players have a higher probability of landing on the orange properties than that of the other two reds. Since the probability of a player being in Jail is highest, it makes sense then that some of the most landed on properties would be the average roll of two dice, being the orange properties. For reference, a chart with the probabilities in ascending order is given by Figure 6 in the Appendix.

Another longer and more accurate way of modeling Monopoly is by taking each space and dividing it up into three parts. For the spaces other than Jail, the three states are reached without doubles, reached with one throw of doubles, and reached with two throws of doubles. The three states in Jail were the three chances to get out of Jail by rolling doubles. Thus, there are 120 states and a $120x120$ matrix to solve for (Bewersdorff). This method is complex, and I do not believe my computer could handle it.

Further research into this topic could consist of using these probabilities to calculate the average game time in turns and figuring out which properties have the highest expected value. While Jorg Bewersdorff's analysis of Monopoly in *Luck, Logic, and White Lies: The Mathematics of Games* does calculate the probabilities and expected return, I would like to explore the expected value based on my probability results. From my results, I discovered that Jail is by far the most landed on, with the red property of Illinois Ave coming up second. This further proves that the red properties are a good place to have a monopoly on and that Park Place and Boardwalk might have a high rental cost, but the odds of landing on it are not in your favor. Sometimes the most dangerous player is the one with the most unassuming properties.

Board games are a fun night of strategy and goodhearted competitiveness, unless it's Monopoly of course. Many fail to realize just how based in mathematics the games we play are. Board games have many different variables that affect the outcome of the game. The random chance from rolling the dice and the choice of how to play the game to the player's benefit all have a strong base in mathematics. Many board games can be disassembled and evaluated in

terms of numbers and calculations. In order to further the knowledge of the masses regarding the board game's foundation in mathematics, this thesis analyzed simple games, brought about understanding to sample mathematical scenarios, and further explored Monopoly.
Understanding simple games helps to best understand larger and more complex models, like how understanding the method of the 3-space and 4-space scenario helped to develop a way to create a $43x43$ model. By focusing on Monopoly and calculating the probabilities with the more complex rules, like exiting Jail and doubles, the most probable space a player will land on and the best choices in order to win are found. Overall, this thesis further expanded the knowledge of Monopoly in the mathematical world and helped to open the eyes of those who lacked the knowledge regarding the connection of math and board games. Just remember, using this information is not cheating, so go out, go directly to Jail, and win.

Works Cited

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Appendix

Figure 1: Monopoly Board

Figure 2: 43x43 Movement Matrix

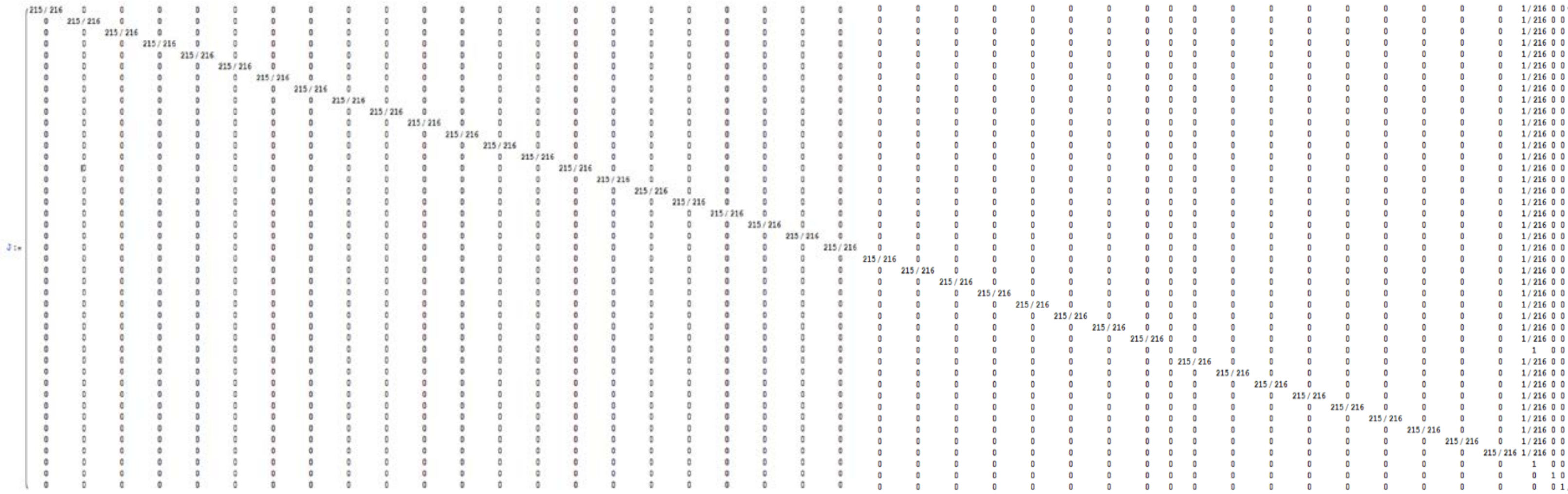

Figure 3: 43x43 Jail Matrix

Figure 4: 43x43 Community Chest Matrix

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\end{pmatrix}
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Figure 5: 43x43 Chance Matrix

State	Space Name	Probability (%)
30	Policeman	0
36	Chance	.813
7	Chance	.824
22	Chance	1.03
2	Community Chest	1.80
1	Mediterranean Ave	2.03
38	Luxury Tax	2.05
37	Park Place	2.06
3	Baltic Ave	2.07
6	Oriental Ave	2.15
10	Visiting Jail	2.16
9	Connecticut Ave	2.19
8	Vermont Ave	2.20
13	States Ave	2.20
33	Community Chest	2.21
4	Income Tax	2.22
17	Community Chest	2.30
34	Pennsylvania Ave	2.34
15	Pennsylvania Railroad	2.38
29	Marvin Gardens	2.40
14	Virginia Ave	2.44
32	North Carolina Ave	2.45
27	Ventnor Ave	2.48
31	Pacific Ave	2.49
39	Boardwalk	2.49
12	Electric Company	2.50
26	Atlantic Ave	2.50
23	Indiana Ave	2.52
35	Short Line	2.55
21	Kentucky Ave	2.57
11	St. Charles Place	2.58
42	Behind Bars (Second Turn)	2.63
16	St. James Place	2.69
28	Water Works	2.76
19	New York Ave	2.78
20	Free Parking	2.80
18	Tennessee Ave	2.81
5	Reading Railroad	2.83
25	B. & O. Railroad	2.85
0	Go	2.93
24	Illinois Ave	2.95
41	Behind Bars (First Turn)	3.15
40	Behind Bars	3.78

Figure 6: Final Probabilities in Decreasing Order