Minimal Surfaces, Soap Films, and the Weierstrass-Enneper Representation

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Abstract: Throughout the study of mathematics, one will find that it is divided into many different subjects of study. Frequently, one will see that these subjects overlap and cover the same material, which is both exciting and useful. No more so is this overlapping effect seen, than in the study of minimal surfaces and soap films. Within this area of study: complex analysis, differential geometry, physics, and multivariable calculus converge together to bring upon the study of minimal surfaces, their connection to soap films, and finally their parameterizations through the Weierstrass-Enneper representation. Through the course of this paper we will discover a link between differential geometry and complex analysis by studying isothermal coordinates, harmonic and analytic functions, and their motivation towards the Weierstrass-Enneper representation. Furthermore, the link between minimal surfaces and surfaces that minimize area will also be made with regards to soap films.

Introduction

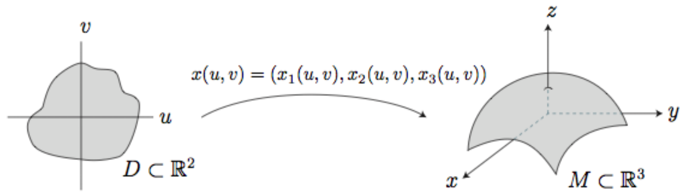
To begin, I would like to first describe what a minimal surface is as well as the different types that are studied. Throughout this paper the surfaces that will be studied will be in $\mathbb{R}^3$. Minimal surfaces minimize area locally, and can be thought of as surfaces where at each point on the surface the bending of the surface upwards is matched by the bending of the surface downwards in the orthogonal direction. This bending is described by curvature, which we will define later. There are two types of minimal surfaces that we will discuss in this paper: complete and embedded. A complete minimal surface is boundaryless and an embedded minimal surface has no self-intersections. Examples of complete embedded minimal surfaces are the plane, catenoid, helicoid, and Scherk's doubly periodic surface.

Section 1: Differential Geometry

We will first review material from differential geometry, beginning with parameterizations of surfaces in $\mathbb{R}^3$.

1.1 Parameterizations

Every point on a surface in $\mathbb{R}^3$can be represented as an ordered-triple, $(x, y, z) \in \mathbb{R}^3$. It can also be represented by two parameters. This can be done by letting D be an open set in $\mathbb{R}^2$, in which case the surface M can be represented by the function $\mathbf{x}(u, v) = \big(x_1(u, v), x_2(u, v), x_3(u, v)\big)$ where $\mathbf{x}: D \to \mathbb{R}^3$ and M is the image of $\mathbf{x}(D)$. It is important to note that $\mathbf{x}$ must be differentiable. This implies that every $x_k(u, v)$ has a continuous partial derivative of every order in the open set D. If these properties are sufficed, then we have a **parameterization x**.



Parameterization of Surface [Dor110]

A very important concept to understand is that when we are dealing with a function of one variable, $y = F(x)$, we want it to satisfy the horizontal line test. Furthermore, $F(x)$ is a one-dimensional object in $\mathbb{R}^2$, and it is parameterized by the mapping of $u \to (u, F(u))$ which takes $\mathbb{R} \to \mathbb{R}^2$. This idea can be taken one dimension higher for a function of two variables, $z = f(x, y)$. Now, (x, y) is two-dimensional and f satisfies the vertical line test where it is parallel to the z-axis. Going further, the graph of $z = f(x, y)$ is a two-dimensional surface in $\mathbb{R}^3$ where its height is z at the point (x, y) **[Dor113]**.

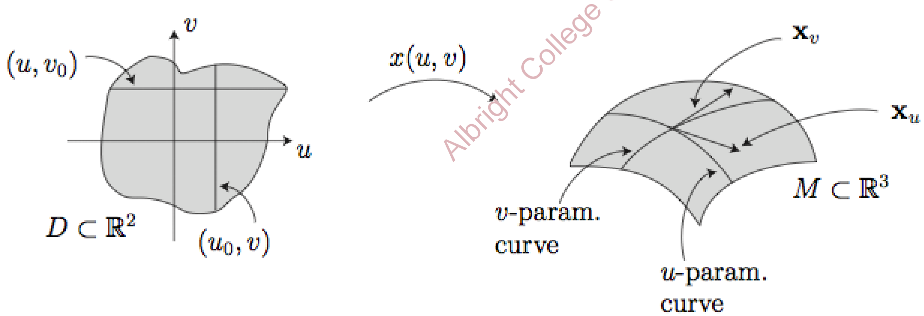
1.2 Tangent Planes and Normal Vectors

Let $\mathbf{x}(u, v)$ be a parameterization of the surface $M \subset \mathbb{R}^3$, then if $v = v_0$ is fixed and u is allowed to vary, then $\mathbf{x}(u, v_0)$ only depends on a single parameter. This is defined as a ***u*-parameter curve**. Likewise, if $u = u_0$ is fixed and v is allowed to vary, then we have a ***v*-parameter curve**.

Now we can define tangent vectors for our u, v-parameter curves. These are given by

$$\mathbf{x}_u = \left(\frac{\partial x_1}{\partial u}, \frac{\partial x_2}{\partial u}, \frac{\partial x_3}{\partial u}\right), \qquad \mathbf{x}_v = \left(\frac{\partial x_1}{\partial v}, \frac{\partial x_2}{\partial v}, \frac{\partial x_3}{\partial v}\right)$$

and for any point $p = \mathbf{x}(u_0, v_0)$ on the surface M, there will be two vectors $\mathbf{x}_u|_{(u_0,v_0)}$ and $\mathbf{x}_v|_{(u_0,v_0)}$. For any parameterization we will require that $\mathbf{x}_u$ and $\mathbf{x}_v$ are linearly independent, which results in the span of these two vectors giving us a plane. This plane is defined as the **tangent plane.** Note that $\mathbf{x}_u$ and $\mathbf{x}_v$ are linearly independent if and only if their cross product is zero.

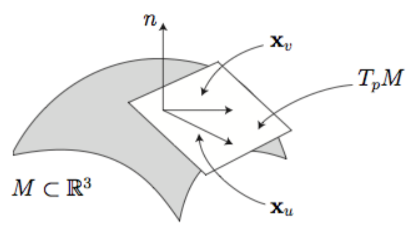


***u*-parameter curve and *v*-parameter curve [Dor115]**

Definition 1.2.1: The **unit normal** vector to a surface M at the point $p = \mathbf{x}(a, b)$ is defined by

$$\hat{\boldsymbol{n}}(a, b) = \left.\frac{\mathbf{x}_u \times \mathbf{x}_v}{|\mathbf{x}_u \times \mathbf{x}_v|}\right|_{(a,b)}$$

A surface that does not have a well-defined unit normal over the whole surface is called a **non-orientable surface.** An example of this is the Mobius strip.



The unit normal of a surface [Dor116]

If the surface M is an orientable surface, then there are two unit normal vectors, one that points inward and one outward, for any point $p \in M$.

1.3 Curvature

Any plane that contains $\hat{\boldsymbol{n}}$ must intersect the surface in a curve called α. This curve is very special to our study of minimal surfaces as it allows us to measure curvature. **Curvature** is the measure of how this curve bends away from the tangent plane at the point p.

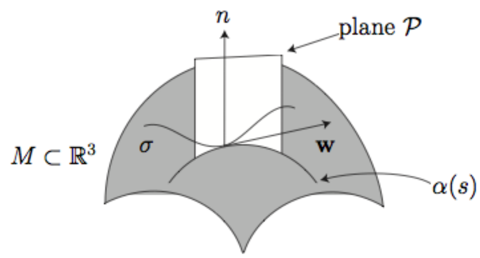
A curve in $\mathbb{R}^3$can be parameterized by a single variable function, $\alpha(t)$ where $\alpha: [a, b] \to \mathbb{R}^3$. However, this parameterization is not unique to the function. For example, the unit circle can be parameterized by both, $\alpha(t) = (\mathrm{r}\cos u\,, r\sin u)$ and $\alpha(t) = (\mathrm{r}\sin u, \mathrm{r}\cos u)$. We can simplify our parameterizations by requiring that they be of unit speed. A curve is of **unit speed** if $|\alpha'(t)|$=1, $\forall t$. If $\alpha(t)$ is regular or smooth, (i.e. $\alpha'(t) \neq \vec{\mathbf{0}}, \forall t \in [a, b]$) but not of unit speed, then we can first parameterize the curve by arc length to have a curve of unit speed.

Essentially, **curvature** is the rate of change of the tangent vector at p. Hence, what we are really concerned in knowing is $\hat{\boldsymbol{n}}$, the rate at which the tangent vectors vary. So, the **curvature of a unit speed curve** is given by, $\frac{\partial}{\partial s}\,\alpha'(s) = \alpha''(s)$.

1.4 Normal and Mean Curvature

Now that we have an understanding of normal and tangent vectors as well as the notion of curvature, we can work towards a definition for normal and mean curvature. As we will soon see, mean curvature lies at the foundation of the study of minimal surfaces. Both normal and mean curvature will also be vital in defining the first and second fundamental forms.

Suppose that we are given a curve $\sigma(s)$ on a surface M. In which case, we can compute the tangent vector $\vec{\boldsymbol{w}}$ of the curve $\sigma(s)$ at the point $p \in M$. Likewise, $\vec{\boldsymbol{w}} \times \hat{\boldsymbol{n}}$ will create the plane P where the intersection of the plane with the surface will be the curve $\alpha(s)$, as shown below.



Normal Curvature [Dor122]

Definition 1.4.1: The **normal curvature** in the $\vec{\boldsymbol{w}}$ direction is defined by

$$k(\vec{\boldsymbol{w}}) = \alpha''(s) \cdot \hat{\boldsymbol{n}}$$

We can then conclude that the normal curvature is essentially the measure of how much M bends towards the unit normal vector as you approach p in the direction of the tangent vector. Furthering this idea, by rotating the plane about the unit normal vector, this will result in a set of curves, each of which has their own respective curvature. Hence, each direction with have its own normal curvature and tangent vector.

Denote k_1 and k_2 as the maximum and the minimum normal curvatures of the set curves at the point p. These are called the **principal curvatures**. Furthermore, these principle curvatures define the directions in which the normal curvature attains both its absolute maximum and absolute minimum value, which are called the **principal directions**.

Definition 1.4.2: The **mean curvature**, denoted H, of a surface M at the point p is given by

$$H = \frac{k_1 + k_2}{2}$$

where k_1 and k_2 are the principle curvatures.

It is important to notice that when $k_j > 0$, this corresponds to a bending towards the unit normal vector, while $k_j < 0$, results in a bending away from the unit normal vector.

1.5 The First and Second Fundamental Forms

In order to discuss a definition for a minimal surface in terms of its mean curvature we need to be able to compute H at every point p on the surface M explicitly, rather than being plagued with computing the principle curvatures at every single point.

In order to define an explicit formula for mean curvature, consider a curve α which is of unit speed. In which case we have,

$$|\alpha'| = 1 \Leftrightarrow |\alpha'|^2 = 1$$

$$\Leftrightarrow 1 = \alpha' \cdot \alpha'$$

$$= (\mathbf{x}_u \partial u + \mathbf{x}_v \partial v) \cdot (\mathbf{x}_u \partial u + \mathbf{x}_v \partial v)$$

$$= \mathbf{x}_u \cdot \mathbf{x}_u \partial u^2 + 2\mathbf{x}_u \mathbf{x}_v \partial u \partial v + \mathbf{x}_v \cdot \mathbf{x}_v \partial v^2$$

$$= E\partial u^2 + 2F\partial u \partial v + G\partial v^2 \qquad \textbf{(First Fundamental Form)}$$

where $E = \mathbf{x}_u \cdot \mathbf{x}_{u,}$ $F = \mathbf{x}_u \cdot \mathbf{x}_v$, and $G = \mathbf{x}_v \cdot \mathbf{x}_v$. We call $\boldsymbol{E}, \boldsymbol{F}, \boldsymbol{G}$ **the coefficients of the first fundamental form**.

Next, we will derive the second fundamental form using the definition of the normal curvature for the curve α. Since $\alpha' \perp \hat{\boldsymbol{n}}$, $\alpha' \cdot \hat{\boldsymbol{n}} = 0$. Hence, taking the derivative on both sides gives,

$$(\alpha' \cdot \hat{\boldsymbol{n}})' = 0 \Rightarrow \alpha'' \cdot \hat{\boldsymbol{n}} + \alpha' \cdot \hat{\boldsymbol{n}}' = 0$$

$$\Leftrightarrow \alpha'' \cdot \hat{\boldsymbol{n}} = -\alpha' \cdot \hat{\boldsymbol{n}}'$$

Similarly, we also have that $-\mathbf{x}_u \cdot \hat{\boldsymbol{n}}_u = \mathbf{x}_{uu} \cdot \hat{\boldsymbol{n}}, -\mathbf{x}_u \cdot \hat{\boldsymbol{n}}_v = \mathbf{x}_{uv} \cdot \hat{\boldsymbol{n}},\ -\mathbf{x}_v \cdot \hat{\boldsymbol{n}}_v = \mathbf{x}_{vv} \cdot \hat{\boldsymbol{n}}$. Using the previous equivalence and the definition for the normal curvature of the curve, we have

$$k(\vec{\boldsymbol{w}}) = \alpha'' \cdot \hat{\boldsymbol{n}}$$

$$= -\alpha' \cdot \hat{\boldsymbol{n}}'$$

$$= -(\mathbf{x}_u du + \mathbf{x}_v dv) \cdot (\hat{\boldsymbol{n}}_u du + \hat{\boldsymbol{n}}_v dv)$$
$$= -\mathbf{x}_u \cdot \hat{\boldsymbol{n}}_u du^2 - (\mathbf{x}_u \cdot \hat{\boldsymbol{n}}_v + \mathbf{x}_v \cdot \hat{\boldsymbol{n}}_u) du dv - \mathbf{x}_v \cdot \hat{\boldsymbol{n}}_v dv^2$$

$$= \mathbf{x}_{uu} \cdot \hat{\boldsymbol{n}}\, du^2 + 2\, \mathbf{x}_{uv} \cdot \hat{\boldsymbol{n}}\, du dv + \mathbf{x}_{vv} \cdot \hat{\boldsymbol{n}}\, dv^2$$

$$= e\, du^2 + 2f du dv + g\, dv^2 \qquad \textbf{(Second Fundamental Form)}$$

where $e = \mathbf{x}_{uu} \cdot \hat{\boldsymbol{n}}$, $f = \mathbf{x}_{uv} \cdot \hat{\boldsymbol{n}}$, and $g = \mathbf{x}_{vv} \cdot \hat{\boldsymbol{n}}$. Similarly, to the coefficients of the first fundamental form, these are called the **coefficients of the second fundamental form**. They represent the amount in which the surface is bending away from the tangent plane **[Dor124]**.

<u>Section 2: Minimal Surfaces</u>

2.1 The Minimal Surface Equations

In the introduction we said that the curvature bending upwards on a minimal surface was matched by curvature bending downwards in the orthogonal direction. Essentially, these curvatures cancel each other out, resulting in our mean curvature, $H = 0$. This suggests the following definition.

Definition 2.1.1: A surface M is a **minimal surface** if $H = 0$ for every point $p \in M$.

However, the problem of calculating the curvature at every point arises. We can use the coefficients of the first and second fundamental forms in order to develop an equation for H. Setting H equal to zero will determine when our surface of study is indeed a minimal surface. Following the derivation of the formula for H in terms of the coefficients of the first and second fundamental forms, we will use the Monge patch to derive the Minimal Surface Equation. A Monge patch is a parameterization of a graph, which is a function of the two variables u and v, where it is parameterized by $\mathbf{x}(u,v) = \big(u, v, f(u,v)\big)$.

We will first prove the following theorem.

Theorem 2.1.2: $H = \frac{Eg+Ge-2Ff}{2(EG-F^2)}$.

Proof: We will follow the approach taken by **[Opr40].**

Let $\overrightarrow{\boldsymbol{w_1}}$ and $\overrightarrow{\boldsymbol{w_2}}$ be two orthogonal unit vectors tangent to the surface where k_1 and k_2 are their normal curvatures. Now using the curves, $\alpha_1(s) = \big(u_1(s), v_1(s)\big)$ and $\alpha_2(s) = \big(u_2(s), v_2(s)\big)$, and also letting $p_1 = du_1 + idu_2$ and $p_2 = dv_1 + idv_2$.

Now using the second fundamental form and also plugging k_1 and k_2 we have,

$$\begin{aligned} 2H = k_1 + k_2 = \ & k(\overrightarrow{\boldsymbol{w_1}}) + k(\overrightarrow{\boldsymbol{w_2}}) \\ &= e\,du_1^2 + 2f du_1 dv_1 + g\,dv_1^2 + e\,du_2^2 + 2f du_2 dv_2 + g\,dv_2^2 \\ &= e(du_1^2 + du_2^2) + 2f(du_1 dv_1 + du_2 dv_2) + g(dv_1^2 + dv_2^2) \\ &= e(p_1\overline{p_1}) + f(p_1\overline{p_2} + \overline{p_1}p_2) + g(p_2\overline{p_2}) \end{aligned}$$

Next we want to eliminate p_1 and p_2 from the equation. This will allow us to move closer to having a formula for H in terms of our fundamental form coefficients. Recall that $1 = \alpha' \cdot \alpha' = |\alpha|^2 = Edu^2 + 2Fdudv + Gdv^2$ and since $\overrightarrow{\boldsymbol{w_1}}$ and $\overrightarrow{\boldsymbol{w_2}}$ are perpendicular, $\overrightarrow{\boldsymbol{w_1}} \cdot \overrightarrow{\boldsymbol{w_2}} = \alpha_1'(s) \cdot \alpha_2'(s) = 0$.

$$\Rightarrow E du_1 du_2 + F(du_1 dv_2 + du_2 dv_1) + G dv_1 dv_2 = 0 \qquad \textbf{(Eq. 1)}$$

Now,

$$\begin{aligned} &Ep_1^2 + 2Fp_1p_2 + Gp_2^2 \\ &\quad = E[du_1^2 - du_2^2 + i2du_1du_2] + 2F[du_1dv_1 - du_2dv_2 + i(du_1dv_2 + du_2dv_1)] \\ &\qquad + G[dv_1^2 - dv_2^2 + i2dv_1dv_2] \\ &\quad = 2i[Edu_1du_2 + F(du_1dv_2 + du_2dv_1) + Gdv_1dv_2] + [Edu_1^2 + 2Fdu_1dv_1 + Gdv_1^2] - \\ &\qquad [Edu_2^2 + 2Fdu_1dv_2 + Gdv_2^2] \\ &\quad = 0 + 1 - 1 \qquad \text{(By Eq. 1)} \\ &\quad = 0 \end{aligned}$$

By the quadratic formula we have that, $p_1 = \left(-\frac{F}{E} \pm i\frac{\sqrt{EG-F^2}}{E}\right)p_2$ and $\overline{p_1} = \left(-\frac{F}{E} \mp i\frac{\sqrt{EG-F^2}}{E}\right)\overline{p_2}$ and,

$$p_1\overline{p_1} = \left(\frac{F^2}{E^2} + EG - F^2E\right)p_2\overline{p_2} = \frac{G}{E}p_2\overline{p_2} \qquad \textbf{(Eq. 2)}$$

$$p_1\overline{p_2} + \overline{p_2}p_2 = -\frac{2F}{E}p_2\overline{p_2} \qquad \textbf{(Eq. 3)}$$

Now, substituting back our original equation,

$$2H = k_1 + k_2 = e(du_1^2 + du_2^2) + 2f(du_1dv_1 + du_2dv_2) + g(dv_1^2 + dv_2^2)$$
$$= e(p_1\overline{p_1}) + f(p_1\overline{p_2} + \overline{p_1}p_2) + g(p_2\overline{p_2})$$
$$= e\left(\frac{G}{E}p_2\overline{p_2}\right) + f\left(-\frac{2F}{E}p_2\overline{p_2}\right) + g(p_2\overline{p_2})$$
$$= \left[e\frac{G}{E} + f\left(\frac{-2F}{E}\right) + g\right]p_2\overline{p_2} \qquad \textbf{(Eq. 4)}$$

Now all that is left to do is to get rid of $p_2\overline{p_2}$ from the previous equation. Recall that,

$$1 = \alpha' \cdot \alpha' = Edu^2 + 2Fdudv + Gdv^2$$

Hence, using the above equation we have,

$$2 = E(du_1^2 + du_2^2) + 2F(du_1dv_1 + du_2dv_2) + G(dv_1^2 + dv_2^2)$$
$$= E(p_1\overline{p_1}) + F(p_1\overline{p_2} + \overline{p_1}p_2) + G(p_2\overline{p_2})$$
$$= E\left(\frac{G}{E}p_2\overline{p_2}\right) + F\left(\frac{-2F}{E}p_2\overline{p_2}\right) + Gp_2\overline{p_2}$$
$$\Rightarrow 2 = \left[2G - \frac{2F^2}{E}\right]p_2\overline{p_2}$$
$$\Leftrightarrow p_2\overline{p_2} = \frac{2}{2G - \frac{2F^2}{E}}$$
$$= \frac{E}{EG - F^2}$$

Now plugging into Eq. 4 we have

$$2H = k_1 + k_2 = e(du_1^2 + du_2^2) + 2f(du_1dv_1 + du_2dv_2) + g(dv_1^2 + dv_2^2)$$
$$= e(p_1\overline{p_1}) + f(p_1\overline{p_2} + \overline{p_1}p_2) + g(p_2\overline{p_2})$$
$$= \left[e\frac{G}{E} + f\left(\frac{-2F}{E}\right) + g\right]p_2\overline{p_2}$$
$$= \left[e\frac{G}{E} + f\left(\frac{-2F}{E}\right) + g\right]\frac{E}{EG - F^2}$$
$$\Leftrightarrow H = \frac{Eg + Ge - 2Ff}{2(EG - F^2)} \qquad \blacksquare$$

This formula allows us to tell whether a surface is minimal by computing the first and second fundamental form coefficients and plugging into the equation. Now that we have a formula for the mean curvature in terms of the coefficients of the fundamental form, let's derive the Minimal Surface Equation. This will give a condition when a Monge patch is minimal.

To do this we will need to consider the Monge patch. Now, we will state the theorem which we seek to prove.

Theorem 2.1.3 (The Minimal Surface Equation): A surface M defined by the parameterization, $\mathbf{x}(u,v) = (u, v, f(u,v))$, is minimal if and only if $f_{uu}(1 + f_v^2) - 2f_uf_vf_{uv} + f_{vv}(1 + f_u^2) = 0$.

Proof: Let $z = f(x, y)$ be a function of two variables and its graph is defined by the Monge patch, $\mathbf{x}(u, v) = \big(u, v, f(u, v)\big)$. Hence, by computing relevant partial derivatives, the unit normal vector, and all of our fundamental form coefficients we have,

$$\mathbf{x}_u = (1, 0,\ f_u) \qquad \mathbf{x}_{uu} = (0, 0, f_{uu}) \qquad \mathbf{x}_{vv} = (0, 0, f_{vv})$$

$$\boldsymbol{x}_v = (0, 1, f_v) \qquad \mathbf{x}_{uv} = (0, 0, f_{uv}) \qquad \mathbf{x}_u \times \boldsymbol{x}_v = (-f_u, -f_v, 1)$$

$$\hat{\boldsymbol{n}} = \frac{\mathbf{x}_u \times \boldsymbol{x}_v}{\|\mathbf{x}_u \times \boldsymbol{x}_v\|} = \frac{1}{\sqrt{1 + f_u^2 + f_v^2}}(-f_u, -f_v, 1)$$

$$E = \mathbf{x}_u \cdot \mathbf{x}_u = 1 + f_u^2 \qquad F = \mathbf{x}_u \cdot \mathbf{x}_v = f_u f_v \qquad G = \mathbf{x}_v \cdot \mathbf{x}_v = 1 + f_v^2$$

$$e = \mathbf{x}_{uu} \cdot \hat{\boldsymbol{n}} = \frac{f_{uu}}{\sqrt{1+f_u^2+f_v^2}} \qquad f = \ \mathbf{x}_{uv} \cdot \hat{\boldsymbol{n}} = \frac{f_{uv}}{\sqrt{1+f_u^2+f_v^2}} \qquad g = \mathbf{x}_{vv} \cdot \hat{\boldsymbol{n}} = \frac{f_{vv}}{\sqrt{1+f_u^2+f_v^2}}$$

Now, from Theorem 2.1.2 we proved that $H = \frac{Eg+Ge-2Ff}{2(EG-F^2)}$ and by definition 2.1.1 we defined a surface to be minimal if $H = 0$. Hence,

$$H = 0 \Leftrightarrow 0 = \frac{Eg+Ge-2Ff}{2(EG-F^2)}$$

$$= \frac{(1+f_u^2)\frac{f_{vv}}{\sqrt{1+f_u^2+f_v^2}} + (1+f_v^2)\frac{f_{uu}}{\sqrt{1+f_u^2+f_v^2}} - 2f_u f_v \frac{f_{uv}}{\sqrt{1+f_u^2+f_v^2}}}{2\Big((1+f_u^2)(1+f_v^2) - (f_u f_v)^2\Big)}$$

$$= \frac{(1+f_u^2)f_{uu} + (1+f_v^2)f_{vv} - 2f_u f_v f_{uv}}{2\left(1+f_u^2+f_v^2\right)^{3/2}}$$

Whereby, we see that this will be zero when the numerator is equal to zero, and hence we have that a surface parameterized by $\mathbf{x}(u, v) = \big(u, v, f(u, v)\big)$ will be minimal if and only if

$$f_{uu}(1 + f_v^2) - 2f_u f_v f_{uv} + f_{vv}(1 + f_u^2) = 0 \qquad \blacksquare$$

Note that the minimal surface equation is for a minimal surfaces that is described by a function of two variables, and that it is not an inclusive formula to tell whether any particular surface is minimal.

2.2 Minimal Surfaces Parameterizations

Now that we have established formulas for checking whether a surface is minimal, we can define several parameterizations for surfaces that are easily shown to be minimal using the previous two theorems, and ones that we will represent via soap films later on in the paper. We will also come to see how these parameterizations arise through the use of the Weierstrass-Enneper representation.

Plane: $\mathbf{x}(u, v) = (u, v, 0)$

Enneper's Surface: $\mathbf{x}(u, v) = (u - \frac{u^3}{3} + uv^2, v - \frac{v^3}{3} + u^2 v,\ u^2 - v^2)$

Catenoid: $\mathbf{x}(u, v) = (\mathrm{a} \cosh v \cos u\,, a \cosh v \sin u\,, av)$

Helicoid: $\mathbf{x}(u,v) = (\text{a} \sinh v \cos u\,, a \sinh v \sin u\,, au)$

Scherk's doubly periodic surface: $\mathbf{x}(u,v) = \left(u, v, \ln\left(\frac{\cos u}{\cos v}\right)\right)$

Scherk's singly periodic surface: $\mathbf{x}(u,v) = (\operatorname{arcsinh}(u)\,, \operatorname{arcsinh}(v)\,, \arcsin\,(uv))$

2.3 Isothermal Parameterizations

So far in our study of minimal surfaces we have seen that a minimal surface defined by a function of two variables must satisfy the minimal surface equation given in Theorem 2.1.3. Furthermore, we saw that solutions to this second order partial differential equation gave us minimal surfaces. Continuing on, we will find that by using what is called an isothermal parameterization, we can simplify our parameterizations and move towards the Weierstrass-Enneper representation.

Definition 2.3.1: An **isothermal parameterization** is a parameterization $\mathbf{x}(u,v)$ where $E = \mathbf{x}_u \cdot \boldsymbol{x}_u = \mathbf{x}_v \cdot \boldsymbol{x}_v = G$ and $F = \mathbf{x}_u \cdot \boldsymbol{x}_v = 0$.

We saw in Section 1.5 that the coefficients of the first fundamental form describe the distortion of the lengths on the surface. Thus, we see that since an isothermal parameterization has $F = 0$ we see that $\mathbf{x}_u \perp \boldsymbol{x}_v$, and since $E = G$ we see that the distortion of the lengths is of the same factor in orthogonal directions.

An important theorem to note, but one which will not be proven is that every minimal surface in $\mathbb{R}^3$ has an isothermal parameterization. We will soon see how important an isothermal parameterization is in our proofs of theorems leading up to the Weierstrass-Enneper representation. But first, we prove the following theorem.

Theorem 2.3.2: Let M be a surface with an isothermal parameterization. Then M is minimal if and only if $e = -g$.

Proof: Let M be a surface with an isothermal parameterization. Hence, $E = G$ and $F = 0$.

$$\Rightarrow 0 = H = \frac{Eg + Ge - 2Ff}{2(EG - F^2)} \Leftrightarrow 0 = \frac{E(g+e)}{2(E^2)} \Leftrightarrow 0 = \frac{(g+e)}{2(E)} \Leftrightarrow e = -g$$

Hence, we see that M is minimal if and only if $e = -g$. ■

2.4 Complex Analysis and Conjugate Minimal Surfaces

Now we will introduce concepts from complex analysis. We will discuss conjugate minimal surfaces as well as set up the foundations for the Weierstrass-Enneper representation.

Fundamental to complex analysis is the study of analytic functions.

Definition 2.4.1: A complex function $f(z)$ is said to be **analytic** or **holomorphic** at every point z_0 if $\lim_{h\to 0}\frac{f(z_0+h)-f(z_0)}{h}$ exists.

What this is really requiring is that the complex function is differentiable at every point in the specified region. Furthermore, we know that if $f(z) = x(u,v) + iy(u,v)$ is an analytic function then the Cauchy-Riemann equations hold. This is stated in the following theorem.

Theorem 2.4.2: Given an analytic function, the **Cauchy-Riemann equations** hold, and are given by

$$x_u = y_v \quad \text{and} \quad x_v = -y_u$$

If the Cauchy-Riemann equations hold, then x and y are called **harmonic conjugates.**

The concept of analytic functions will allow us to relate a minimal surface to what is known as its conjugate minimal surface.

Definition 2.4.3: Let **x** and **y** be isothermal parameterizations of minimal surfaces such that their component functions are pairwise harmonic conjugates, in which case we have

$$\mathbf{x}_{\boldsymbol{u}} = \mathbf{y}_{\boldsymbol{v}} \qquad \mathbf{x}_{\boldsymbol{v}} = -\mathbf{y}_{\boldsymbol{u}}$$

Then **x** and **y** are called **conjugate minimal surfaces.**

A very interesting example of conjugate minimal surfaces is the catenoid and the helicoid. In Section 2.2, the parameterizations for these two surfaces were given. This relation can easily be shown by starting with the parameterization of the catenoid, using the Cauchy-Riemann equations and reverse partial differentiate the parameterization until you arrive at the parameterization of the helicoid.

2.5 Harmonic Functions and Minimal Surfaces

In the previous section we discussed how complex analysis allowed us to define conjugate minimal surfaces through the use of the Cauchy-Riemann equations. In this section, we will discover the link between harmonic functions, isothermal parameterizations and the Laplace Operator.

Definition 2.5.1: The **Laplace Operator** is given by $\Delta\,\mathbf{x} = \mathbf{x}_{\boldsymbol{uu}} + \mathbf{x}_{\boldsymbol{vv}}$.

To begin, we will first prove a theorem which defines the necessary and sufficient condition in which a surface is minimal as well as see the relationship between the Laplace Operator and the mean curvature under an isothermal parameterization. We will also define a harmonic function.

Definition 2.5.2: A real-valued function $\mathbf{x}(u,v)$ is **harmonic** if its second-order partial derivatives are continuous and $\Delta\,\mathbf{x} = \mathbf{x}_{\boldsymbol{uu}} + \mathbf{x}_{\boldsymbol{vv}} = 0$.

Theorem 2.5.3: If the parameterization **x** is isothermal, then $\Delta\,\mathbf{x} = \mathbf{x}_{\boldsymbol{uu}} + \mathbf{x}_{\boldsymbol{vv}} = (2EH)\,\hat{\boldsymbol{n}}$**.**

In order to complete this proof we will need to first discuss the **Christoffel symbols**, which are also called the **acceleration formulas**. When we were discussing the tangent vectors for our u, v-parameter curves, denoted $\mathbf{x}_{\boldsymbol{u}}$ and $\mathbf{x}_{\boldsymbol{v}}$, we saw that since they were linearly independent, $\{\mathbf{x}_{\boldsymbol{u}}, \mathbf{x}_{\boldsymbol{v}}\}$ formed a basis for the tangent plane. Furthermore, since $\hat{\boldsymbol{n}}$ is normal to the tangent plane, we quickly see that $\{\mathbf{x}_{\boldsymbol{u}}, \mathbf{x}_{\boldsymbol{v}}, \hat{\boldsymbol{n}}\}$ forms a basis for $\mathbb{R}^3$. Now, the acceleration formulas express the fundamental accelerations: $\mathbf{x}_{\boldsymbol{uu}}$, $\mathbf{x}_{\boldsymbol{uv}}$, and $\mathbf{x}_{\boldsymbol{vv}}$ in terms of this basis. Assuming that $\boldsymbol{F} = 0$, we have that following orthogonal decomposition,

$$\mathbf{x}_{\boldsymbol{uu}} = \Gamma_{uu}^{u}\mathbf{x}_{\boldsymbol{u}} + \Gamma_{uu}^{v}\mathbf{x}_{\boldsymbol{v}} + e\hat{\boldsymbol{n}}$$

$$\mathbf{x}_{\boldsymbol{uv}} = \Gamma_{uv}^{u}\mathbf{x}_{\boldsymbol{u}} + \Gamma_{uv}^{v}\mathbf{x}_{\boldsymbol{v}} + f\hat{\boldsymbol{n}}$$

$$\mathbf{x}_{\boldsymbol{vv}} = \Gamma_{vv}^{u}\mathbf{x}_{\boldsymbol{u}} + \Gamma_{vv}^{v}\mathbf{x}_{\boldsymbol{v}} + g\hat{\boldsymbol{n}}$$

Now we simply solve for each Γ. This is done by taking the dot product with respect to the upper index of each Γ, whereby we use the orthogonal properties of the dot product, as well as the definition of the second fundamental form coefficients to simplify the expression. After doing so, we have the following **acceleration formulas,**

$$\mathbf{x}_{\boldsymbol{uu}} = \frac{E_u}{2E}\mathbf{x}_{\boldsymbol{u}} - \frac{E_v}{2G}\mathbf{x}_{\boldsymbol{v}} + e\hat{\boldsymbol{n}}$$

$$\mathbf{x}_{\boldsymbol{uv}} = \frac{E_v}{2E}\mathbf{x}_{\boldsymbol{u}} + \frac{G_u}{2G}\mathbf{x}_{\boldsymbol{v}} + f\hat{\boldsymbol{n}}$$

$$\mathbf{x}_{\boldsymbol{vv}} = \frac{-G_u}{2E}\mathbf{x}_{\boldsymbol{u}} + \frac{G_v}{2G}\mathbf{x}_{\boldsymbol{v}} + g\hat{\boldsymbol{n}}$$

We are now in a position to prove Theorem 2.5.3.

Proof: Assume that the parameterization is isothermal. Hence, $E = G$ and $F = 0$**.**

$$\begin{aligned}\mathbf{x}_{\boldsymbol{uu}} + \mathbf{x}_{\boldsymbol{vv}} &= \left(\frac{E_u}{2E}\mathbf{x}_{\boldsymbol{u}} - \frac{E_v}{2G}\mathbf{x}_{\boldsymbol{v}} + e\hat{\boldsymbol{n}}\right) - \left(\frac{-G_u}{2E}\mathbf{x}_{\boldsymbol{u}} + \frac{G_v}{2G}\mathbf{x}_{\boldsymbol{v}} + g\hat{\boldsymbol{n}}\right)\\ &= (e+g)\,\hat{\boldsymbol{n}}\\ &= 2E\left(\tfrac{e+g}{2E}\right)\hat{\boldsymbol{n}}\\ &= 2EH\hat{\boldsymbol{n}} \qquad \text{Since if } \mathbf{x} \text{ is isothermal.}\end{aligned}$$

■

From this we see that for a minimal surface M, $\Delta\,\mathbf{x} = \mathbf{x}_{\boldsymbol{uu}} + \mathbf{x}_{\boldsymbol{vv}} = 0$, which implies that a surface is minimal if and only if the Laplacian is equal to 0. We state the following theorem.

Theorem 2.5.4: A surface with an isothermal parameterization $\mathbf{x}(u,v) = \big(x_1(u,v), x_2(u,v), x_3(u,v)\big)$ is minimal if and only if $x_1, x_2,$ and x_3 are harmonic functions.

Proof: Assume M is a surface with an isothermal parameterization.

($\Longrightarrow$) Assume M is minimal. This implies $H = 0$. Hence, by the previous theorem we know that

$\mathbf{x}_{\boldsymbol{uu}} + \mathbf{x}_{\boldsymbol{vv}} = 0$. Hence, the coordinate functions of $\mathbf{x}(u,v)$ are harmonic.

($\Longleftarrow$) Assume if $x_1, x_2,$ and x_3 are harmonic functions. Hence, we know that $\mathbf{x}_{uu} + \mathbf{x}_{vv} = 0$. Furthermore, we then have that $\mathbf{x}_{uu} + \mathbf{x}_{vv} = (2EH)\hat{\boldsymbol{n}} = 0$. Now, since $\hat{\boldsymbol{n}} \neq 0$ and $\boldsymbol{E} = \mathbf{x}_u \cdot \boldsymbol{x}_u \neq 0$ since if it were 0, then the surface would only be a point, which implies that $H = 0$, which gives us that M is a minimal surface. ∎

As we will soon see, the Weierstrass-Enneper representation gives us the connection between complex analysis and minimal surfaces, which we anticipated in the introduction. Moreover, we will see that we can construct minimal surfaces by looking at analytic functions in a specific parameterized formula.

Section 3: The Weierstrass-Enneper Representation

As we will soon see the Weierstrass-Enneper representation allows us to create minimal surfaces by choosing analytic functions. But first we will use complex analysis, in particular analytic and harmonic functions, to work towards this representation.

Let M be a minimal surface with an isothermal parameterization. Now let $\boldsymbol{z} = u + iv$ be a complex coordinate. From complex analysis, we know that $\boldsymbol{z} = u + iv$ and its conjugate, $\bar{\boldsymbol{z}} = u - iv$ can be solved for u and v, where we then have, $u = \frac{\boldsymbol{z}+\bar{\boldsymbol{z}}}{2}$ and $v = \frac{\boldsymbol{z}-\bar{\boldsymbol{z}}}{2i}$. In which case we can then write the parameterization is terms of $\boldsymbol{z}$ and $\bar{\boldsymbol{z}}$ as,

$$\mathbf{x}(z,\bar{z}) = \big(x_1(z,\bar{z}), x_2(z,\bar{z}), x_3(z,\bar{z})\big)$$

Furthermore, we have the following notation, $\phi = (\varphi_1, \varphi_2, \varphi_3)$ where,

$$\varphi_k = \frac{\partial x_k}{\partial z} = \frac{1}{2}\left(\frac{\partial x_k}{\partial u} - i\frac{\partial x_k}{\partial v}\right) \qquad (\varphi_k)^2 = \frac{1}{4}\left[\left(\frac{\partial x_k}{\partial u}\right)^2 - \left(\frac{\partial x_k}{\partial v}\right)^2 - 2i\frac{\partial x_k}{\partial u}\frac{\partial x_k}{\partial v}\right]$$

$$(\phi)^2 = (\varphi_1)^2 + (\varphi_2)^2 + (\varphi_3)^2 \qquad \mathbf{x}_u \cdot \boldsymbol{x}_u = \sum_{k=1}^{3}\left(\frac{\partial x_k}{\partial u}\right)^2$$

$$|\phi|^2 = |\varphi_1|^2 + |\varphi_2|^2 + |\varphi_3|^2 \qquad \mathbf{x}_v \cdot \boldsymbol{x}_v = \sum_{k=1}^{3}\left(\frac{\partial x_k}{\partial v}\right)^2$$

Now if we denote the complex function $f(u,v) = x(u,v) + iy(u,v)$, we can also define the function f in terms of $\boldsymbol{z}$ and $\bar{\boldsymbol{z}}$. Using the chain rule, we can derive the following formulas,

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial x}{\partial u} + \frac{\partial y}{\partial v}\right) + \frac{i}{2}\left(\frac{\partial y}{\partial u} - \frac{\partial x}{\partial v}\right)$$

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial x}{\partial u} - \frac{\partial y}{\partial v}\right) + \frac{i}{2}\left(\frac{\partial y}{\partial u} + \frac{\partial x}{\partial v}\right)$$

We will show the derivation of the first, and leave out the derivation for the second as it follows from the first.

$$\frac{\partial f}{\partial z} = \left(\frac{\partial f}{\partial u}\frac{\partial u}{\partial z}\right) + \left(\frac{\partial f}{\partial v}\frac{\partial v}{\partial z}\right) = \frac{1}{2}\left(\frac{\partial x}{\partial u} + i\frac{\partial y}{\partial u}\right) + \frac{1}{2i}\left(\frac{\partial x}{\partial v} + \frac{\partial y}{\partial v}\right) = \frac{1}{2}\left(\frac{\partial x}{\partial u} + \frac{\partial y}{\partial v}\right) + \frac{i}{2}\left(\frac{\partial y}{\partial u} - \frac{\partial x}{\partial v}\right)$$

This results in the following theorem.

Theorem 3.1.1: The complex function f is analytic if and only if $\frac{\partial f}{\partial \bar{\mathbf{z}}} = 0$.

Proof: ($\Longrightarrow$) Assume the complex function f is analytic.

$\Rightarrow \mathbf{x}_{\boldsymbol{u}} = \mathbf{y}_{\boldsymbol{v}}$ and $\mathbf{x}_{\boldsymbol{v}} = -\mathbf{y}_{\boldsymbol{u}}$

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}(\mathbf{x}_{\boldsymbol{u}} - \mathbf{y}_{\boldsymbol{v}}) + \frac{i}{2}(\mathbf{y}_{\boldsymbol{u}} + \mathbf{x}_{\boldsymbol{v}})$$

$$= \frac{1}{2}(\mathbf{y}_{\boldsymbol{v}} - \mathbf{y}_{\boldsymbol{v}}) + \frac{i}{2}(\mathbf{y}_{\boldsymbol{u}} - \mathbf{y}_{\boldsymbol{u}})$$

$$= 0$$

($\Longleftarrow$) Assume $\frac{\partial f}{\partial \bar{z}} = 0$. Hence, we have then that the real and imaginary parts must equal 0.

$\Rightarrow \frac{1}{2}(\mathbf{x}_{\boldsymbol{u}} - \mathbf{y}_{\boldsymbol{v}}) = 0$ and $\frac{1}{2}(\mathbf{y}_{\boldsymbol{u}} + \mathbf{x}_{\boldsymbol{v}}) = 0$

$\Leftrightarrow \mathbf{x}_{\boldsymbol{u}} = \mathbf{y}_{\boldsymbol{v}}$ and $\mathbf{x}_{\boldsymbol{v}} = -\mathbf{y}_{\boldsymbol{u}}$

$\Rightarrow f$ is analytic ■

This theorem provides us with a very easy test to determine whether our complex function is analytic. It tells us that our complex function is analytic if and only if it can be defined only in terms of $\boldsymbol{z} = u + iv$. Now we will prove several theorems in a row that will be used as foundations for the Weierstrass-Enneper representation.

Theorem 3.1.2: $4\left(\frac{\partial}{\partial \boldsymbol{z}}\left(\frac{\partial f}{\partial \bar{\boldsymbol{z}}}\right)\right) = f_{uu} + f_{vv}$

Proof: $4\left(\frac{\partial}{\partial z}\left(\frac{\partial f}{\partial \bar{z}}\right)\right) = 2\left[\frac{\partial}{\partial u}\left(\frac{\partial f}{\partial \bar{z}}\right) - i\frac{\partial}{\partial v}\left(\frac{\partial f}{\partial \bar{z}}\right)\right]$

$$= 2\left[\frac{\partial}{\partial u}\left(\frac{1}{2}(\mathbf{x}_{\boldsymbol{u}} - \mathbf{y}_{\boldsymbol{v}}) + \frac{i}{2}(\mathbf{y}_{\boldsymbol{u}} + \mathbf{x}_{\boldsymbol{v}})\right) - i\frac{\partial}{\partial v}\left(\frac{1}{2}(\mathbf{x}_{\boldsymbol{u}} - \mathbf{y}_{\boldsymbol{v}}) + \frac{i}{2}(\mathbf{y}_{\boldsymbol{u}} + \mathbf{x}_{\boldsymbol{v}})\right)\right]$$

$$= 2\left[\frac{1}{2}\mathbf{x}_{uu} - \frac{1}{2}\mathbf{y}_{vu} + \frac{i}{2}\mathbf{y}_{uu} + \frac{i}{2}\mathbf{x}_{uv}\right] - 2i\left[\frac{1}{2}\mathbf{x}_{uv} - \frac{1}{2}\mathbf{y}_{vv} + \frac{i}{2}\mathbf{y}_{uv} + \frac{i}{2}\mathbf{x}_{vv}\right]$$

$$= \mathbf{x}_{uu} + \mathbf{x}_{vv} + i(\mathbf{y}_{uu} + \mathbf{y}_{vv})$$

$$= f_{uu} + f_{vv}$$ ■

Theorem 3.1.3a: Let M be a surface with parameterization $\mathbf{x}(u,v) = \big(x_1(u,v), x_2(u,v), x_3(u,v)\big)$ and let $\phi = (\varphi_1, \varphi_2, \varphi_3)$ The parameterization $\mathbf{x}(u,v)$ is isothermal iff $(\phi)^2 = (\varphi_1)^2 + (\varphi_2)^2 + (\varphi_3)^2 = 0$.

Proof: $(\phi)^2 = (\varphi_1)^2 + (\varphi_2)^2 + (\varphi_3)^2$

$$= \frac{1}{4}\sum_{k=1}^{3}\left[\left(\frac{\partial x_k}{\partial u}\right)^2 - \left(\frac{\partial x_k}{\partial v}\right)^2 - 2i\frac{\partial x_k}{\partial u}\frac{\partial x_k}{\partial v}\right]$$

$$= \frac{1}{4}\big((\mathbf{x}_u \cdot \mathbf{x}_u) - (\mathbf{x}_v \cdot \mathbf{x}_v) - 2i(\mathbf{x}_u \cdot \mathbf{x}_v)\big)$$

$$= \frac{1}{4}[E - G - 2iF]$$

Hence, we see that the parameterization will be isothermal if and only if $E = G, F = 0 \iff (\phi)^2 = 0$. ■

Theorem 3.1.3b: Let M be a surface with parameterization $\mathbf{x}(u,v) = \big(x_1(u,v), x_2(u,v), x_3(u,v)\big)$ and let $\phi = (\varphi_1, \varphi_2, \varphi_3)$ where $\varphi_k = \frac{\partial x_k}{\partial z}$. If $\mathbf{x}(u,v)$ is isothermal, then M is minimal if and only if each φ_k is analytic. **[Dor145]**

Proof: Assume $\mathbf{x}(u,v)$ is isothermal. Now by Theorem 2.5.4 we know that for each k, x_k is harmonic if and only if φ_k is analytic and by the previous two theorems we have,

$$f_{uu} + f_{vv} = \frac{\partial^2 x_k}{\partial u \partial u} + \frac{\partial^2 x_k}{\partial v \partial v} = 4\left(\frac{\partial}{\partial \bar{z}}\left(\frac{\partial x_k}{\partial z}\right)\right) = 4\left(\frac{\partial}{\partial \bar{z}}(\varphi_k)\right) = 0$$

Which results since an isothermal parameterization is minimal if and only if the Laplacian equals 0. Therefore f is analytic. ■

We have been working with the variable x_k, which is defined in terms of the two variables $\boldsymbol{z}$ and $\bar{\mathbf{z}}$. In order to resolve this problem we can solve $\varphi_k = \frac{\partial x_k}{\partial z}$ for x_k, which will give us a representation of a single variable. The following derivation is adapted from **[Dor145]**.

Since x_k is a function of two variables we have, $dx_k = \frac{\partial x_k}{\partial u}du + \frac{\partial x_k}{\partial v}dv$ and $dz = du + idv$.

$$\varphi_k\, dz = \frac{\partial x_k}{\partial z}\, dz = \frac{1}{2}\left(\frac{\partial x_k}{\partial u} - i\frac{\partial x_k}{\partial v}\right)(du + idv)$$

$$= \frac{1}{2}\left[\frac{\partial x_k}{\partial u}du + \frac{\partial x_k}{\partial v}dv + i\left(\frac{\partial x_k}{\partial u}dv - \frac{\partial x_k}{\partial v}du\right)\right]$$

Similarly, we have for the conjugate,

$$\overline{\varphi_k\, dz} = \overline{\varphi_k}\,\overline{dz} = \overline{\frac{\partial x_k}{\partial z}}\,\overline{dz} = \frac{1}{2}\left(\frac{\partial x_k}{\partial u} + i\frac{\partial x_k}{\partial v}\right)(du - idv)$$

$$= \frac{1}{2}\left[\frac{\partial x_k}{\partial u}du + \frac{\partial x_k}{\partial v}dv - i\left(\frac{\partial x_k}{\partial u}dv - \frac{\partial x_k}{\partial v}du\right)\right]$$

Adding the two together we have,

$$2Re(\varphi_k\, dz) = \varphi_k\, dz + \overline{\varphi_k\, dz} = \frac{\partial x_k}{\partial u}du + \frac{\partial x_k}{\partial v}dv = dx_k$$

And by integrating, we have $x_k = 2Re\int \varphi_k\, dz + c_k$, but we can ignore the constants since they do not affect the geometry of the surface because c_k only relates to a translation of the image and the 2 scales the surface. Therefore we can reduce our coordinate function to $x_k = Re\int \varphi_k\, dz$. To bring all of this together, if we have the situation where φ_k for $k = 1, 2, 3$ are analytic functions where $(\phi)^2 = 0$ and $|(\phi)^2| \neq 0$ is finite, then the parameterization

$$\mathbf{x} = \left(Re\int \varphi_1(z)\, dz,\ Re\int \varphi_2(z)\, dz,\ Re\int \varphi_3(z)\, dz\right)$$

will define a minimal surface. We now have a method to construct a minimal surface. All we have to do is find $\phi = (\varphi_1, \varphi_2, \varphi_3)$ such that $(\phi)^2 = 0$. To do this we start with an analytic function f, a meromorphic function g, and also require that fg^2 is also analytic, then define

$$\varphi_1 = f(1 - g^2) \qquad \varphi_2 = if(1 + g^2) \qquad \varphi_3 = 2fg$$

We can readily check to see that $(\phi)^2 = 0$.

$$(\phi)^2 = (\varphi_1)^2 + (\varphi_2)^2 + (\varphi_3)^2 = f^2(1 - 2g^2 + g^4) - f^2(1 + 2g^2 + g^4) + 4f^2g^2 = 0$$

Lemma 3.1.4: For any complex-valued functions f and g on any domain, and

$\phi = (f(1 - g^2),\ if(1 + g^2),\ 2fg)$, the following are true.

1) $(Re\ \phi)^2 - (Im\ \phi)^2 = 0$
2) $(Re\ \phi) \cdot (Im\ \phi) = 0$

Proof: Note that we previously showed that $(\phi)^2 = 0$. Hence, $0 = Re(\phi)^2 = (Re\ \phi)^2 - (Im\ \phi)^2$

since $Re(u + iv)^2 = u^2 - v^2$. Likewise, for (2) we have, $0 = Im(\phi)^2 = 2(Re\ \phi) \cdot (Im\ \phi)$

since $Im(u + iv)^2 = 2uv$. ■

Theorem 3.1.5: The Weierstrass-Enneper Representation for Minimal Surfaces: If $f(z)$ is analytic on a domain D, $g(z)$ is meromorphic on D, and fg^2 is analytic on D, then the parameterization

$\mathbf{x}(z, \bar{z}) = \big(x_1(z, \bar{z}), x_2(z, \bar{z}), x_3(z, \bar{z})\big)$ where

$$x_1(z,\bar{z}) = Re\int \varphi_1 dz \quad x_2 = Re\int \varphi_2 dz \quad x_3 = Re\int \varphi_3 dz$$

defines a minimal surface.

Proof 1: [Sha54] Let $\Phi = \int(\varphi_1, \varphi_2, \varphi_3)dz$ denote the integral of ϕ. Now note the following,

$$\mathbf{x}_u = Re[\Phi_u] = Re\left[\frac{d\Phi}{dz}\frac{\partial z}{\partial u}\right] = Re\phi$$

since $z = u + iv$, which results $\frac{\partial z}{\partial u} = 1$.

$$\mathbf{x}_v = Re[\Phi_v] = Re\left[\frac{d\Phi}{dz}\frac{\partial z}{\partial v}\right] = Re(i\phi) = -Im\phi$$

Now by Lemma 3.1.4 we have that $\mathbf{x}_u \cdot \mathbf{x}_v = 0$, $\mathbf{x}_u{}^2 = \mathbf{x}_v{}^2$, and $\mathbf{x}_{uu} = Re(\phi')$ and $\mathbf{x}_{vv} = -Re(\phi')$. Therefore we have $\mathbf{x}_{uu} + \mathbf{x}_{vv} = 0$. This tells us that our parameterization defines a minimal surface. ■

Example 3.1.6: Weierstrass-Enneper representation for Enneper's surface

Let $f = 1$ and $g = z$. Then, we have the following parameterization,

$$\mathbf{x}(z,\bar{z}) = \left(Re\int(1-z^2)\,dz, \quad Re\int i(1+z^2)dz, \quad Re\int 2z\,dz\right)$$

$$= \left(Re\left(z - \tfrac{1}{3}z^3\right),\ Re\left(i\left(z + \tfrac{1}{3}z^3\right)\right),\ Re(z^2)\right)$$

Now letting $z = u + iv$ yields

$$\mathbf{x}(u,v) = \left(u - \frac{1}{3}u^3 + uv^2, \quad v + u^2v - \frac{v^3}{3}, \quad u^2 - v^2\right)$$

which is the parameterization that we defined for Enneper's surface in Section 2.2.

Now suppose that in the Weierstrass-Enneper representation we assume that g is analytic and that its inverse function exists. Furthermore, let g be a function variable and define $\tau = g$ where $d\tau = g'dz$. We will also define $F(\tau) = \frac{f}{g'}$ and we see that,

$$F(\tau)d\tau = F(\tau)g'dz = \frac{f}{g'}g'dz = fdz$$

So, substituting g with τ and fdz with $F(\tau)d\tau$ we can define the second form of the Weierstrass-Enneper representation.

Theorem 3.1.7: For every analytic function $F(\tau)$, a minimal surface is defined by the parameterization $\mathbf{x}(z,\bar{z}) = \left(x_1(z,\bar{z}), x_2(z,\bar{z}), x_3(z,\bar{z})\right)$ where

$$x_1(z,\bar{z}) = Re\int F(\tau)(1-\tau^2)dz \quad x_2 = Re\int iF(\tau)(1+\tau^2)dz \quad x_3 = Re\int 2\tau F(\tau)dz$$

defines a minimal surface.

Now we have a method for obtaining a minimal surface, which only requires one analytic function instead of two. This representation tells us that any analytic function will result in a minimal surface, but the computation of these integrals is not always as simple.

As we did with the Weierstrass-Enneper representation, we will show the derivation for the parameterization of Enneper's surface using the second form.

Example 3.1.8: Enneper's Surface: Let $F(\tau) = 1$,

$$\mathbf{x}(z, \bar{z}) = \left(Re\int (1-\tau^2)dz\,, Re\int (i+i\tau^2)dz, Re\int 2i\tau\, dz \right)$$

$$= \left(Re\left(\tau - \frac{1}{3}\tau^3\right), Re\left(i\tau + \frac{i}{3}\tau^3\right), Re(\tau^2) \right)$$

Now since g was assumed to be a complex function we have, $\tau = g = u + iv$, which will yield the parameterization from Section 2.2.

$$= \left(Re\left(u + iv - \frac{1}{3}(u+iv)^3\right), Re\left(i(u+iv) + \frac{i}{3}(u+iv)^3\right), Re((u+iv)^2) \right)$$

$$\mathbf{x}(u, v) = \left(u - \frac{1}{3}u^3 + uv^2, \qquad v + u^2v - \frac{v^3}{3}, \qquad u^2 - v^2 \right)$$

To summarize, we have been studying minimal surfaces, and while doing so want to look at isothermal parameterizations in terms of a two-variable real-valued function. We found that this parameterization was connected to harmonic functions and resulted in the Laplacian equaling 0. Furthermore, we saw that an isothermal parameterization of a harmonic function gave us the required complex analytic functions that we needed to construct the Weierstrass-Enneper representation for minimal surfaces. As such, we found a connection between differential geometry and complex analysis.

Section 4: The Connection to Soap Films

While studying minimal surfaces, one will find that it is sometimes hard to visualize the surfaces. A solution to this is soap films, which are physical models of minimal surfaces. This is just as amazing as it is unexpected.

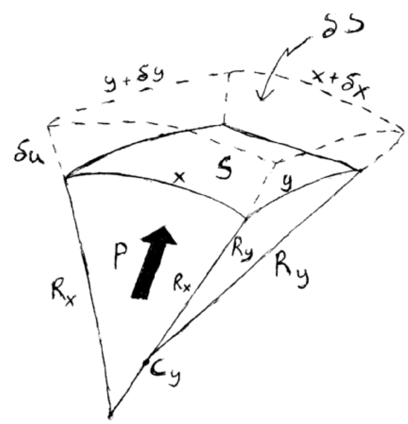
Section 4.1: Surface Tension

The key to soap films is surface tension. **Surface tension** is the force per length on a liquid, which is given by $\sigma = \frac{F}{l}$. Within a liquid the molecules exert forces on each other that are of equal strength. The deeper the molecules are located within the liquid, the more they feel a force of equal magnitude from every direction. However, a molecule near the surface will feel a stronger force from the molecules within the liquid than it will from those near the surface. As such, any molecule near the surface will be pulled into the liquid, and will result in a curvature along a boundary **[Oprea1]**. This tautness created within the liquid is the surface tension.

Now using the following theorem we can show that a soap film is a physical model of a minimal surface.

Theorem 4.1.1: Every soap film is a physical model of a minimal surface. **[Oprea13]**

Proof: Consider a small section of a soap film, which is expanded a small increment by an internal pressure that has been increased. As shown, in the diagram below.



Now work is defined as the force per distance and surface tension gives a soap film the potential to do work, so we see that the amount of work done by expanding the film is,

$$W = F \cdot d = p \cdot S \cdot du = p \cdot x \cdot y \cdot du = \sigma \cdot \Delta S$$

where S is the surface area, p is the pressure, and σ is the surface tension. Moreover, the amount of work is the surface tension times the forces per unit length, ΔS.

$$\Delta S = (x + dx)(y + dy) - xy$$

This gives us the ratio between the original radii of the soap film and the expanded radii,

$$\frac{x + dx}{R_1 + du} = \frac{x}{R_1} \qquad \frac{y + dy}{R_2 + du} = \frac{y}{R_2}$$

In which case ΔS becomes,

$$\Delta S = (x + dx)(y + dy) - xy = x\left(1 + \frac{du}{R_1}\right)y\left(1 + \frac{du}{R_2}\right) - xy$$

And if we consider a small increment du, $\Delta S = xy\, du\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$, and the work becomes,

$$pxydu = \sigma \cdot \Delta S = \sigma\, xy\, du\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$$

$$p = \sigma\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$$

The equation $p = \sigma\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$ is called the **Laplace-Young equation**, and it tells us that the pressure difference on either side of the film is given by the product of its surface tension. Now consider the case where a soap film is bounded by a wire, then $H = \frac{k_1+k_2}{2} = \frac{1}{2}\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$. Where $\frac{1}{R_1}, \frac{1}{R_2}$ are the normal curvatures of the surface in perpendicular directions. Since there is no enclosed volume, the pressure much be the same on both sides of the soap film. Hence, $p = 0$, which reduces the Laplace-Young equation to the following.

$$0 = \sigma\left(\frac{1}{R_1} + \frac{1}{R_2}\right) = 2H\sigma$$

Which implies that $H = 0$. Therefore every soap film is a physical model of a minimal surface. ■

The physics of a soap film is determined by surface tension. In this case, a soap film feels a lesser force outside of it then inside of it; otherwise it would implode. This is due to the pressure within the bubble being larger than the pressure outside. Soap films give us representations of minimal surfaces, since the inward force caused by the surface tension, shrinks the surface to the smallest area possible. This gives rise to the following theorem.

Theorem 4.1.2: If a surface is area minimizing, then the surface is minimal. **[Oprea143]**

Proof: Let $z = f(u, v)$ be a function of two variables. The surface area of the surface is given by $\mathcal{L} = \iint_S \sqrt{1 + f_u^2 + f_v^2}\, du dv$ for a Monge patch, and since we are assuming that the surface is area minimizing, we can use the two-variable Euler-Lagrange equation, which gives us the necessary condition for $z = f(u, v)$ to be minimized. Consider the Euler-Lagrange equation for two independent variables.

$$\frac{\partial \mathcal{L}}{\partial f} - \frac{\partial}{\partial x}\frac{\partial \mathcal{L}}{\partial f_u} - \frac{\partial}{\partial y}\frac{\partial \mathcal{L}}{\partial f_v} = 0$$

Now substituting $\mathcal{L}$ into the Euler-Lagrange equation, we have

$$0 = -\frac{\partial}{\partial u}\left(\frac{f_u}{\sqrt{1 + f_u^2 + f_v^2}}\right) - \frac{\partial}{\partial v}\left(\frac{f_v}{\sqrt{1 + f_u^2 + f_v^2}}\right)$$

$$= \frac{f_{uu}(1 + f_u^2 + f_v^2) - f_u(f_u f_{uu} + f_v f_{uv})}{(1 + f_u^2 + f_v^2)^{\frac{3}{2}}} + \frac{f_{vv}(1 + f_u^2 + f_v^2) - f_v(f_u f_{uv} + f_v f_{vv})}{(1 + f_u^2 + f_v^2)^{\frac{3}{2}}}$$

$$= \frac{(1+f_u^2)f_{uu} + (1+f_v^2)f_{vv} - 2f_u f_v f_{uv}}{2(1+f_u^2+f_v^2)^{3/2}}$$

Whereby, we see that this will be zero when the numerator is equal to zero, and hence we have that a surface will be area minimizing if and only if it is minimal. ■

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